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COMPACT ENDOMORPHISMS AND CLOSED IDEALS IN BANACH ALGEBRAS

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ABSTRACT. Every infinite-dimensional Banach algebra with a nonzero compact endomorphism has a proper closed nonzero two-sided ideal. When the algebra is commutative, the ideal is also an ideal in the multiplier algebra.

Suppose that A is a Banach algebra with a compact nonzero endomorphism. We show that A has a proper closed nonzero two-sided ideal, and we prove a slightly stronger result when A is commutative. There are many Banach algebras which have compact endomorphisms [**5**]; this can even happen for commutative radical integral domains [**3**, §5]. On the other hand, some primitive Banach algebras, such as the algebra of compact operators on Hilbert space, have no nonzero two-sided ideals. Our results show that algebras with no nonzero closed ideals can have no compact nonzero endomorphisms.

Our proofs involve applying extensions, which we developed in [**4**], of Lomonosov's invariant subspace theorems. Apparently the best that one can do with the original versions of Lomonosov's theorem [**6**, **7**] is to show that if A has a compact left or right multiplication operator, then it has both left and right nonzero closed ideals. Using very different methods, Esterle [**1**] has recently made significant contributions to the study of the problem of whether every radical Banach algebra has a nonzero closed ideal.

THEOREM 1. *If the infinite-dimensional Banach algebra A has a compact nonzero endomorphism ϕ, then A has a proper closed nonzero ideal.*

PROOF. Let $A^\#$ be the Banach algebra formed by adjoining an identity to A if A has no identity, and let $A^\# = A$ if A has an identity. For each a in $A^\#$, let L_a be the left multiplication operator $L_a x = ax$, and let $\mathscr{L}$ be the algebra of all left multiplication operators L_a on A for a in $A^\#$. Similarly, let $\mathscr{R}$ be the algebra of right multiplication operators. To show that A has a proper closed ideal, we show that $\mathscr{L}$ and $\mathscr{R}$ have a common proper invariant subspace. We do this by showing that the algebras $\mathscr{L}$ and $\mathscr{R}$ satisfy the hypotheses of [**4**, Theorem (3.2), p. 849].

The map $a \to L_a$ is a continuous linear transformation from $A^\#$ to the algebra of bounded operators on A. Hence $\mathscr{L}$ is an operator range algebra in the sense of [**4**, p. 845] (cf. [**2**, §3]). Since ϕ is a homomorphism, we have $\phi L_a = L_{\phi(a)}\phi$ for every a in

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$A^\sharp$, so that $\phi\mathscr{L} \subseteq \mathscr{L}\phi$. Similarly, $\mathscr{R}$ is an operator range algebra with identity and $\phi\mathscr{R} \subseteq \mathscr{R}\phi$, so that $\phi\mathscr{L}\mathscr{R} \subseteq \mathscr{L}\mathscr{R}\phi$. Finally, $\mathscr{L}\mathscr{R} = \mathscr{R}\mathscr{L}$, so it follows from [**4**, Theorem (3.2), p. 849] that $\mathscr{L}$ and $\mathscr{R}$ have a common invariant subspace which must be a closed nonzero proper ideal in A.

THEOREM 2. *Suppose that A is a commutative Banach algebra which contains at least one element a for which the map $x \to ax$ is not multiplication by a scalar. If A has a compact nonzero endomorphism ϕ, then there is a proper closed nonzero subspace of A which is an ideal in the multiplier algebra of A.*

PROOF. As in the proof of Theorem 1, we let $\mathscr{L}$ be the operator range algebra of left multiplications by elements of $A^\sharp$, and we have $\phi\mathscr{L} \subseteq \mathscr{L}\phi$. It then follows from [**4**, Theorem (3.1), p. 848], or [**2**, Theorem 6, p. 61] that the commutant of $\mathscr{L}$ has an invariant subspace. Since the commutant of $\mathscr{L}$ is the multiplier algebra of A, the theorem is proved.

If the algebra A in Theorem 2 is not commutative, we can use [**4**, Theorem (3.2), p. 849] to conclude that A has a proper nonzero closed subspace which is invariant under all operators which are simultaneously right and left multipliers of A. However, such a subspace need not even be an ideal of A, since we can guarantee only that it is closed under multiplication by central elements of A.

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