

COMPACT ENDOMORPHISMS AND CLOSED IDEALS IN BANACH ALGEBRAS

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ABSTRACT. Every infinite-dimensional Banach algebra with a nonzero compact endomorphism has a proper closed nonzero two-sided ideal. When the algebra is commutative, the ideal is also an ideal in the multiplier algebra.

Suppose that A is a Banach algebra with a compact nonzero endomorphism. We show that A has a proper closed nonzero two-sided ideal, and we prove a slightly stronger result when A is commutative. There are many Banach algebras which have compact endomorphisms [5]; this can even happen for commutative radical integral domains [3, §5]. On the other hand, some primitive Banach algebras, such as the algebra of compact operators on Hilbert space, have no nonzero two-sided ideals. Our results show that algebras with no nonzero closed ideals can have no compact nonzero endomorphisms.

Our proofs involve applying extensions, which we developed in [4], of Lomonosov's invariant subspace theorems. Apparently the best that one can do with the original versions of Lomonosov's theorem [6, 7] is to show that if A has a compact left or right multiplication operator, then it has both left and right nonzero closed ideals. Using very different methods, Esterle [1] has recently made significant contributions to the study of the problem of whether every radical Banach algebra has a nonzero closed ideal.

THEOREM 1. *If the infinite-dimensional Banach algebra A has a compact nonzero endomorphism ϕ , then A has a proper closed nonzero ideal.*

PROOF. Let $A^\#$ be the Banach algebra formed by adjoining an identity to A if A has no identity, and let $A^\# = A$ if A has an identity. For each a in $A^\#$, let L_a be the left multiplication operator $L_a x = ax$, and let \mathcal{L} be the algebra of all left multiplication operators L_a on A for a in $A^\#$. Similarly, let \mathcal{R} be the algebra of right multiplication operators. To show that A has a proper closed ideal, we show that \mathcal{L} and \mathcal{R} have a common proper invariant subspace. We do this by showing that the algebras \mathcal{L} and \mathcal{R} satisfy the hypotheses of [4, Theorem (3.2), p. 849].

The map $a \rightarrow L_a$ is a continuous linear transformation from $A^\#$ to the algebra of bounded operators on A . Hence \mathcal{L} is an operator range algebra in the sense of [4, p. 845] (cf. [2, §3]). Since ϕ is a homomorphism, we have $\phi L_a = L_{\phi(a)}\phi$ for every a in

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$A^\#$, so that $\phi\mathcal{L} \subseteq \mathcal{L}\phi$. Similarly, \mathcal{R} is an operator range algebra with identity and $\phi\mathcal{R} \subseteq \mathcal{R}\phi$, so that $\phi\mathcal{L}\mathcal{R} \subseteq \mathcal{L}\mathcal{R}\phi$. Finally, $\mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$, so it follows from [4, Theorem (3.2), p. 849] that \mathcal{L} and \mathcal{R} have a common invariant subspace which must be a closed nonzero proper ideal in A .

THEOREM 2. *Suppose that A is a commutative Banach algebra which contains at least one element a for which the map $x \rightarrow ax$ is not multiplication by a scalar. If A has a compact nonzero endomorphism ϕ , then there is a proper closed nonzero subspace of A which is an ideal in the multiplier algebra of A .*

PROOF. As in the proof of Theorem 1, we let \mathcal{L} be the operator range algebra of left multiplications by elements of $A^\#$, and we have $\phi\mathcal{L} \subseteq \mathcal{L}\phi$. It then follows from [4, Theorem (3.1), p. 848], or [2, Theorem 6, p. 61] that the commutant of \mathcal{L} has an invariant subspace. Since the commutant of \mathcal{L} is the multiplier algebra of A , the theorem is proved.

If the algebra A in Theorem 2 is not commutative, we can use [4, Theorem (3.2), p. 849] to conclude that A has a proper nonzero closed subspace which is invariant under all operators which are simultaneously right and left multipliers of A . However, such a subspace need not even be an ideal of A , since we can guarantee only that it is closed under multiplication by central elements of A .

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