ON HALPERN'S CONJECTURE FOR CLOSED PLANE CURVES

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ABSTRACT. Let c be a smooth closed plane curve given in general position. A bitangent of c is, by definition, a line which is tangent to c at two different points. Let $B(c)$ and $D(c)$ denote the numbers of all bitangents and all double points of c, respectively. We prove here that if c has no inflection points, $B(c) \leq D(c)(2D(c)-1)$. This is the affirmative answer to Halpern's conjecture.

0. Let $c\colon S^1 \to \mathbf{R}^2$ be a smooth closed curve. If c is in general position, then c has a finite number $B(c)$ of *bitangents*, that is, lines in $\mathbf{R}^2$ which are tangent to c at two points. A bitangent T is said to be *exterior* or *interior*, according as in small neighborhoods of the two tangent points of T to c, the curve c lies on the same side or on opposite sides of T. Let $II(c)$ and $I(c)$ denote the numbers of exterior and interior bitangents of c, respectively; $B(c) = II(c) + I(c)$. We consider other two geometric objects; the double points and the inflection points. A smooth plane curve c has generically a finite number $D(c)$ of double points and no triple points, nor more, and a finite number $F(c)$ of inflection points. It is clear that $F(c)$ is even. (See Figure 1.)

Fabricius-Bjerre discovered an equation which $II(c)$, $I(c)$, $D(c)$ and $F(c)$ obey;

$$(1) \qquad II(c) = I(c) + D(c) + F(c)/2.$$

This equation is a necessary, but not sufficient, condition in the following sense; for example, if $D(c) = F(c) = 0$, then c is convex, hence $B(c) = II(c) + I(c) = 0$; in other words, $(II, I, D, F/2) = (1,1,0,0)$ satisfies (1) but cannot be realized as
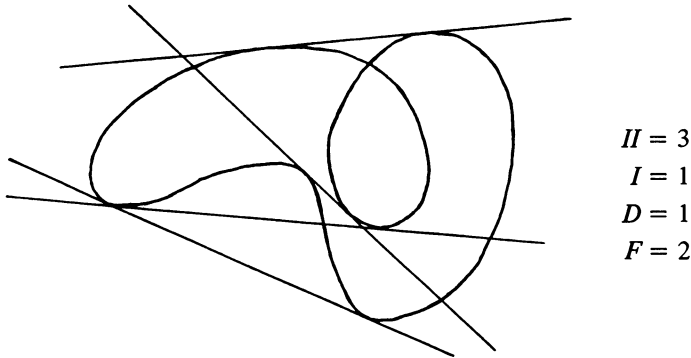

$$II = 3$$
$$I = 1$$
$$D = 1$$
$$F = 2$$

FIGURE 1

General position   Special position   General position
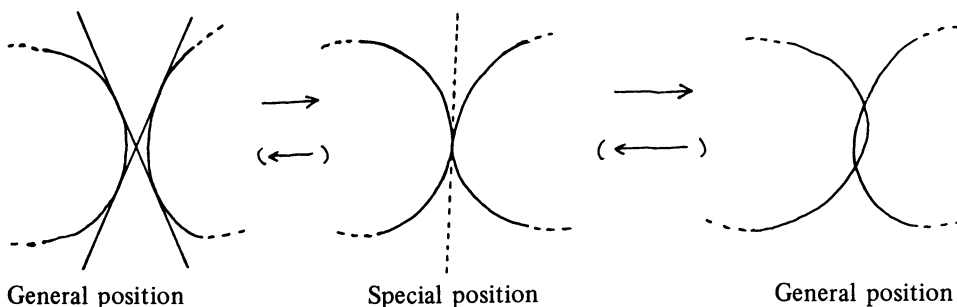
FIGURE 2

$(II, I, D, F/2) = (II(c), I(c), D(c), F(c)/2)$ by some curve c. The question is the following;

Question. What is the necessary and sufficient condition for integers II, I, D, and $F/2$ to be attained by the integers $II(c), I(c), D(c)$ and $F(c)/2$ for some curve c in general position.

Halpern gave this question in [**3**] and conjectured that:

(i) any quadruplet $(II, I, D, F/2)$ satisfying (1) with $F/2 \geq 1$ can be attained by a curve;

(ii) a quadruplet $(II, I, D, F/2)$ with $F = 0$ should satisfy the inequality $I \leq D^2 - D$ and the equality $I \equiv 0 \pmod 2$ besides (1) in order to be attained by a curve.

In that paper Halpern mentioned that general examples show that (i) is true. We will see this in §3. He also said it can be shown that, if $F(c) = 0$, $I(c)$ is even and $I(c) \leq 4D^2(c) + 2D(c)$. The first part is proved by using a deformation argument in the sense of Banchoff (cf. [**1**]), as follows: let c be a given curve with $F(c) = 0$, and let $d(c)$ denote the tangential degree of c. There is a curve c_0 such that $F(c_0) = 0$, $d(c_0) = d(c)$, and $I(c_0)$ is even (for example, the curve $c(l, m, n)$ with $l = m = 0$ and $n = |d(c)| = 1$, defined in §3, Figure 7, has an even number of interior bitangents; in fact, $I(c(0, 0, n)) = n(n - 1)$. c_0 can be deformed to c preserving $F = 0$ in such a controlled way that the changes of I at the special position in the deformation can be observed. The only special position for which the integer I will change is as in Figure 2. The interior bitangents appear or disappear in pairs. It follows that $I(c)$ is even since $I(c_0)$ is even.

Therefore the main part of the conjecture is, if $F(c) = 0$, $I(c) \leq D^2(c) - D(c)$; in other words, *if a smooth closed plane curve c in general position has no inflection points, then the number $B(c)$ of bitangents is less than or equal to $2D^2(c) - D(c)$, where $D(c)$ denotes the number of double points.*

The purpose of this note is to show that *Halpern's conjecture is true*, and to answer the above question in the

THEOREM. *The necessary and sufficient condition for nonnegative integers II, I, D and $F/2$ to be attained by a smooth curve c as*

$$(II, I, D, F/2) = (II(c), I(c), D(c), F(c)/2)$$

is the following:

(i) $II = I + D + F/2$, *and*

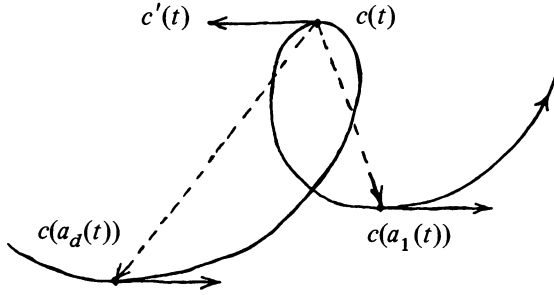(ii) *if $F = 0$, $I \leq D^2 - D$ and I is even.*

FIGURE 3

In §1 we prepare some tools for the proof of $I(c) \leq D^2(c) - D(c)$, which is done in §2. In §3 we construct examples to prove the sufficiency part of the Theorem.

1. In the following, c denotes a smooth closed plane curve parameterized by arc length with only transversal self-intersection and positively curved; $c''(t) = d^2c(t)/dt^2 > 0$, at every point $t \in S^1$, makes sense by identifying the unit circle in $\mathbf{R}^2$ and $\mathbf{R}/2\pi\mathbf{Z}$ in the usual way.

Since $c' = dc/dt\colon S^1 \to S^1$ is a covering map, the number of points t in S^1 such that $c'(t) = z$, for a fixed unit vector $z \in S^1$, is equal to the tangential degree $d = \deg(c)$. Let $a_i\colon S^1 \to S^1$ $(i = 1,\ldots,d)$ be smooth maps such that for each $t \in S^1$,

(2)
$$\begin{cases} -c'(t) = c'(a_i(t)) \quad \text{for any } i, \\ t < a_1(t) < \cdots < a_d(t) < t + 2\pi. \end{cases}$$

Since c has only transversal self-intersection, it follows that $c(t) \neq c(a_i(t))$ for any i and t. Let $g_i\colon S^1 \to S^1$ denote the functions defined by

$$g_i(t) = (c(a_i(t)) - c(t))/\|c(a_i(t)) - c(t)\|,$$

and $f_i\colon S^1 \to S^1$ the functions defined by

$$f_i(t) = c'(t) - g_i(t)$$

for $i = 1,\ldots,d$, where the symbol "$-$" in the definition of f_i makes sense on the additive group S^1. We have

(3) $\deg(f_i) = \deg(c') - \deg(g_i) = d - \deg(g_i)$.

Each (t,i) with $t \in f_i^{-1}(0) \cup f_i^{-1}(\pi)$ corresponds to an interior bitangent, and for each interior bitangent there exist two such pairs (t,i) and (s,j). Moreover, for such a pair (t,i), $dg_i(t)/dt = 0$, and hence $df_i(t)/dt = c''(t) > 0$. Therefore the number of t's with $t \in f_i^{-1}(0) \cup f_i^{-1}(\pi)$ is equal to $2\deg(f_i)$. Thus we obtain the relation between $\deg(f_i)$ and the number $I(c)$ of the interior bitangents as follows:

LEMMA 1. (i) $\deg(f_i) \geq 0$ for $i = 1,\ldots,d$, and (ii) $I(c) = \sum_i \deg(f_i)$.

The images $c'([t, a_1(t)])$ and $c'([a_d(t), t])$ are equal to closed half-circles for each t. Hence, from the mean value theorem, it follows that the angles between $c'(t)$ and $c(a_1(t)) - c(t)$ and between $c'(t)$ and $c(a_d(t)) - c(t)$ are less than π for each t (see Figure 3). This implies

LEMMA 2. $0 = \deg(f_1) = \deg(f_d) = d - \deg(g_1) = d - \deg(g_d)$.
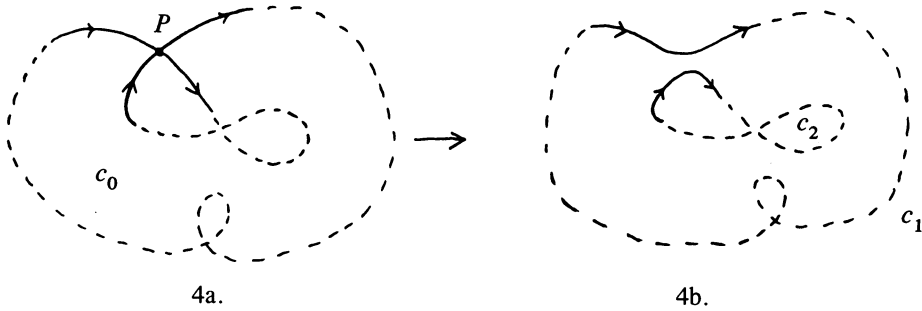
4a.   4b.

FIGURE 4

If c has no self-intersection, then $D(c) = 0$ and c is a convex curve, since $c''(t) > 0$ at every point. Hence $\deg(c) = 1$. In general we have

LEMMA 3. *If c has no triple points (not necessarily $F(c) = 0$), $D(c) \geq |\deg(c)| - 1$.*

PROOF. Suppose there exists a smooth closed curve $c\colon S^1 \to \mathbf{R}^2$ such that $D(c) < |\deg(c)| - 1$, and let c_0 be such a curve with the minimal number $D(c_0)$ of self-intersection points. The equality $D(c) = 0$ implies that c is a Jordan curve and, hence, $|\deg(c)| = 1$. Thus c_0 has at least one self-intersection point p. We replace c_0 by c_1 and c_2 as in Figure 4b. Then we have

$$D(c_0) \geq D(c_1) + D(c_2) + 1,$$

$$|\deg(c_0)| = |\deg(c_1) + \deg(c_2)| \leq |\deg(c_1)| + |\deg(c_2)|.$$

These inequalities and the hypothesis on c_0 imply

$$D(c_1) + D(c_2) < (|\deg(c_1)| - 1) + (|\deg(c_2)| - 1).$$

This and the fact that $D(c_i) < D(c_0)$ for $i = 1$ and 2 contradict the minimality of $D(c_0)$.   Q.E.D.

In the following, we suppose c has no triple points. We define the subset A of $T^2 = S^1 \times S^1$ by $A = \{(t, s) \in T^2; c(t) = c(s)\}$. A is the union of the diagonal set $\Delta \subset T^2$ and $2D(c)$ discrete points which correspond, in pairs, to the self-intersection points of c. Let sym: $T^2 \to T^2$ denote the symmetry of T^2; $\mathrm{sym}(t, s) = (s, t)$. It is clear that $\mathrm{sym}(A) = A$.

To a continuous curve $u = (a, b)\colon S^1 \to T^2 - A$, we associate a map $g_u\colon S^1 \to S^1$ defined by

$$g_u = (c(b(t)) - c(a(t)))/\|c(b(t)) - c(a(t))\|.$$

and define $\mathrm{Deg}(u) = \deg(g_u)$.

LEMMA 4. $\mathrm{Deg}(u)$ *is invariant under homotopies, that is, Deg is a function of the homotopy classes of continuous maps of S^1 to $T^2 - A$.*

PROOF. This is trivial since the degree of $g\colon S^1 \to S^1$ is invariant under homotopies.   Q.E.D.

We consider the residue $\mathrm{Res}(p)$ of each point $p \in A - \Delta$; $\mathrm{Res}(p) = \mathrm{Deg}(u)$, where $u\colon S^1 \to T^2 - A$ is such that u makes a small round at p positively.

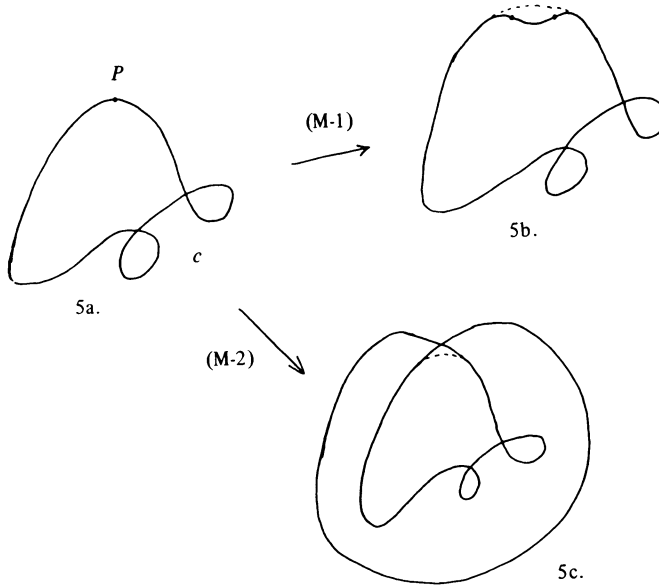LEMMA 5. *For each $p \in A - \Delta$, $\mathrm{Res}(p) = \pm 1$, and $\mathrm{Res}(\mathrm{sym}(p)) = -\mathrm{Res}(p)$.*

FIGURE 5

PROOF. If we take a continuous imbedding $u = (a, b)$: $S^1 \to T^2 - A$ whose image $u(S^1)$ is a sufficiently small square centered at $p \in A - \Delta$, then we easily find that g_u: $S^1 \to S^1$ is a homeomorphism. Hence, $\text{Res}(p) = \pm 1$.

Let v be obtained from u by parallel translation from p to $\text{sym}(p)$ in T^2. Then we have $\bar{a}(t) = b(2\pi - t)$ and $\bar{b}(t) = a(2\pi - t)$, where $v = (\bar{a}, \bar{b})$, and thus $-g_v(t) = g_u(-t)$. Hence we have $\deg(g_v) = \deg(-g_v) = -\deg(g_u)$. This implies $\text{Res}(\text{sym}(p)) = -\text{Res}(p)$.   Q.E.D.

2.

PROPOSITION (HALPERN'S CONJECTURE). *If $F(c) = 0$, then $I(c) \leq D^2(c) - D(c)$.*

PROOF. Suppose $F(c) = 0$ and $c'' > 0$ at every point. The proof for $c'' < 0$ is essentially the same. Let $d = \deg(c)$ and let a_i: $S^1 \to S^1$ $(i = 1, \ldots, d)$ satisfy (2). Define u_i: $S^1 \to T^2 - A$ by $u_i(t) = (t, a_i(t))$. For each i and j with $1 \leq i < j \leq d$, there exists an orientation preserving imbedding h: $S^1 \times [0, 1] \to T^2$ such that $h|_{S^1 \times \{0\}} = u_i$ and $h|_{S^1 \times \{1\}} = u_j$. Hence, by Lemma 5 we have

$$\text{Deg}(u_i) - \text{Deg}(u_j) = - \sum_{p \in h(S^1 \times [0,1])} \text{Res}(p)$$

(4)
$$\leq \text{the number of } p \in A \cap h(S^1 \times [0, 1]) \text{ with } \text{Res}(p) = -1$$
$$\leq D(c).$$

$(I, D, F) = (0, 0, 2)$

$(I, D, F) = (1, 0, 2)$

$(I, D, F) = (2, 0, 2)$
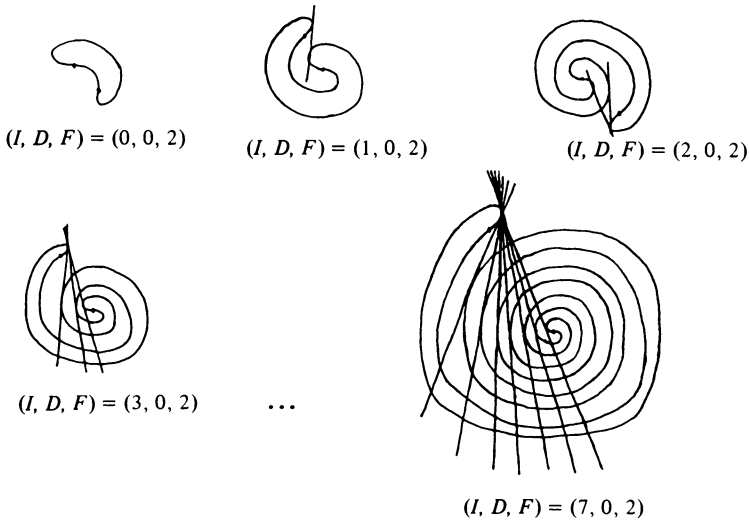
$(I, D, F) = (3, 0, 2)$ $\cdots$

$(I, D, F) = (7, 0, 2)$

FIGURE 6

By Lemma 2 we have $0 = \deg(f_1) = d - \mathrm{Deg}(u_1)$, since $g_1 = g_{u_1}$. Hence, putting $i = 1$, (4) is reduced to

$$(5) \qquad\qquad d - \mathrm{Deg}(u_j) \leq D(c).$$

Thus, from Lemmas 1–3 and inequality (5) it follows that

$$I(c) = \sum_{j=1}^{d} \deg(f_j) = \sum_{j=1}^{d}(d - \mathrm{Deg}(u_j))$$
$$\leq (d-2)D(c) \leq (D(c) - 1)D(c). \quad \text{Q.E.D.}$$

3. Proof of the Theorem (*Sufficiency*). First, we consider two ways to modify a given curve c. Fix a point p on c such that p attains the unique nondegenerate maximum on c of some linear function on $\mathbf{R}^2$. In a small neighborhood of p, we deform c to c_1 as in Figure 5b. Then we have

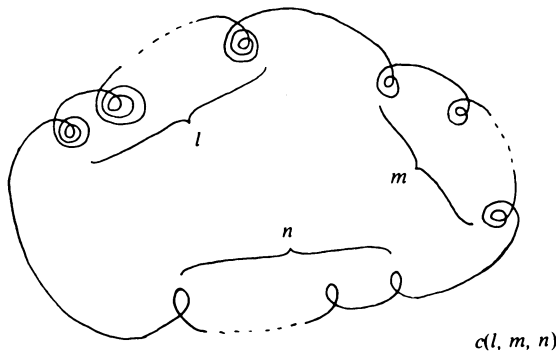$$(I(c_1), D(c_1), F(c_1)/2) = (I(c), D(c), F(c)/2 + 1).$$

$c(l, m, n)$

FIGURE 7

We call this modification (M-1). Next we replace c by c_2 as in Figure 5c. We have

$$(I(c_2), D(c_2), F(c_2)/2) = (I(c), D(c) + 1, F(c)/2).$$

This modification is called (M-2).

(i) The case that c has at least two inflection points: For any nonnegative integer I, there exists a curve c such that $(I(c), D(c), F(c)/2) = (I, 0, 1)$, as the examples in Figure 6 show. Using the modifications (M-1) and (M-2), it follows that for any nonnegative integers I, D and $F/2$ with $F/2 \geq 1$, there exists a curve c' such that

$$(I(c'), D(c'), F(c')/2) = (I, D, F/2).$$

(ii) The case that c has no inflection points: Consider the family of curves $c(l, m, n)$ defined in Figure 7, where l, m and n are nonnegative integers. The numbers I, D and F of $c(l, m, n)$ are as follows;

$$F = F(c(l, m, n)) = 0,$$
$$D = D(c(l, m, n)) = 3l + 2m + n,$$
$$I = I(c(l, m, n)) = D^2 - D - (6l + 2m).$$

We remark that any even integer x, with $0 \leq x < 2D - 2$, is written as $x = 6l + 2m$ for some nonnegative integers l and m with $3l + 2m \leq D$. This implies that for given nonnegative integers D and I, with

$$D^2 - D \geq I > (D - 1)^2 - (D - 1) = D^2 - D - (2D - 2),$$

there exist l, m and n such that

$$(I(c(l, m, n)), D(c(l, m, n)), F(c(l, m, n))) = (I, D, 0).$$

Combining these examples and (M-2), we finish the proof.   Q.E.D.

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