ARRANGEMENTS OF LINES
WITH A LARGE NUMBER OF TRIANGLES

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ABSTRACT. An arrangement of lines is constructed by choosing n diagonals of the regular $2n$-gon. This arrangement is proved to form at least $n(n-3)/3$ triangular cells.

1. Introduction. We shall use the terminology of Grünbaum [5]. By an arrangement $\mathscr{A}$ of lines we mean a finite family of lines $L_1,\ldots,L_n$ in the *real projective* plane Π. The number of lines in $\mathscr{A}$ is denoted by $n(\mathscr{A})$. If no point belongs to more than two of these lines L_i, the arrangement is called *simple*. With an arrangement $\mathscr{A}$ there is associated a 2-dimensional *cell-complex* into which the lines of $\mathscr{A}$ decompose Π. It is well known that in a simple arrangement $\mathscr{A}$ the number of cells (or polygons) of that complex is $(n^2 - n + 2)/2$ $(n = n(\mathscr{A}))$. We shall denote by $p_j(\mathscr{A})$ the number of j-gons among the cells of (the complex associated with) $\mathscr{A}$.

2. Constructions. Let us denote by $P(O)$ a fixed point on the circle $\mathscr{C}$ with centre C. For any real α, let $P(\alpha)$ be the point obtained by rotating $P(O)$ around C, with angle α. Further denote by $L(\alpha)$ the straight line $P(\alpha)\,P(\pi - 2\alpha)$. In case $\alpha \equiv \pi - 2\alpha \pmod{2\pi}$, $L(\alpha)$ is the line tangent to $\mathscr{C}$ at $P(\alpha)$. (See Figure 1.)
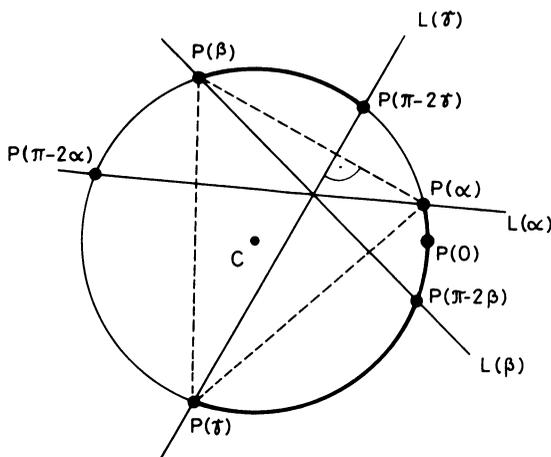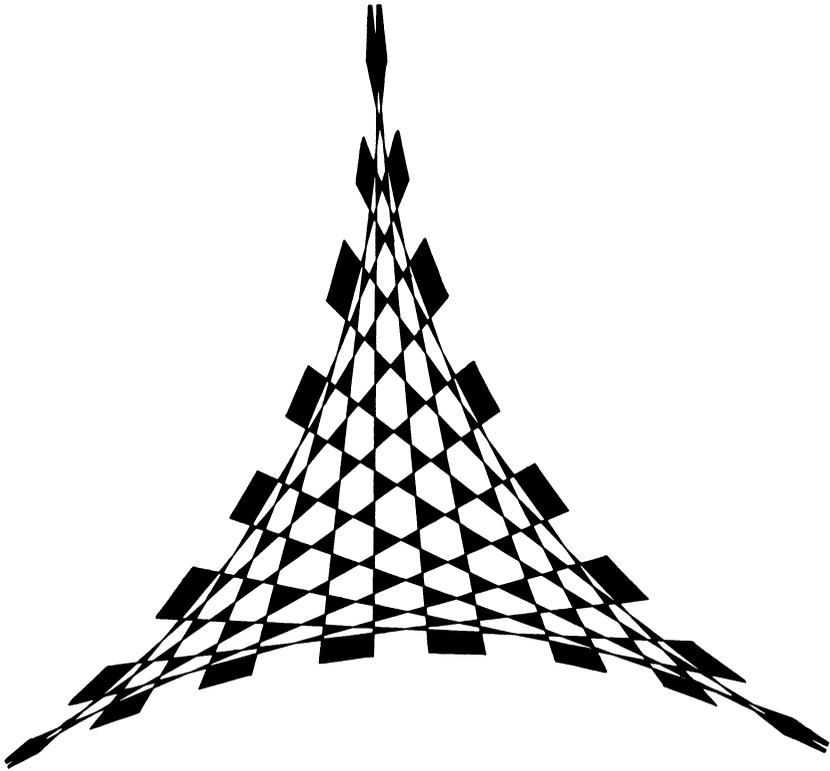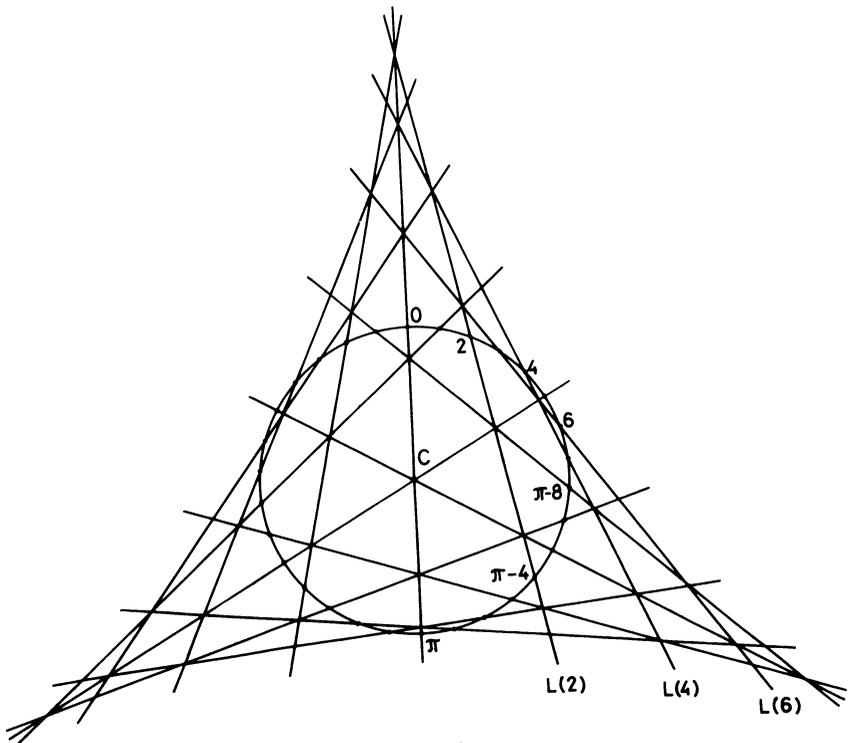

FIGURE 1

FIGURE 2

FIGURE 3

EXAMPLE 1. *Given any integer $n \geqslant 3$, put* $\mathscr{A}_n = \{ L((2i + 1)\pi/n): i = 0, 1, \ldots, n - 1\}$. (See Figure 2.)

EXAMPLE 2. *Given any integer $n \geqslant 3$, put* $\mathscr{B}_n = \{ L(2i\pi/n): i = 0, 1, \ldots, n - 1\}$. (See Figure 3.)

Remark that our set of lines $\{ L(\alpha): 0 \leqslant \alpha < 2\pi \}$ may be regarded as a set of tangents to the arcs of a hypocycloid of third order, drawn in a circle of centre C and radius 3. The line $L(\alpha)$ is tangent to the arc of the cycloid at the αth point. (See Figure 4.) However, we shall not rely upon this fact in what follows.
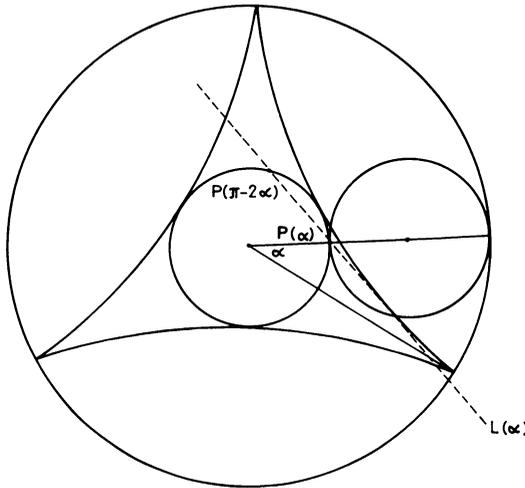

FIGURE 4

LEMMA. *The lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are concurrent if and only if $\alpha + \beta + \gamma \equiv 0$ (mod 2π).*

PROOF. If $\alpha + \beta + \gamma \equiv 0$ (mod 2π), then the sum of lengths of (directed) arcs $(P(\alpha), P(\gamma))$ and $(P(\beta), P(\pi - 2\gamma))$ is equal to π. This implies that $L(\gamma)$ is perpendicular to the line $P(\alpha) P(\beta)$. Hence, the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are the altitudes of the triangle $P(\alpha) P(\beta) P(\gamma)$ (see Figure 1), consequently they meet at one point.

The reverse can be proved similarly.

3. Triangles in a simple arrangement. Grünbaum [5] (cf. Theorem 2.21) pointed out that the maximal number of triangles in a simple arrangement $p_3^s(n) = \max\{ p_3(\mathscr{A}): n(\mathscr{A}) = n, \mathscr{A}$ is simple$\}$ can be estimated by $p_3^s(n) \leqslant \lfloor n(n - 1)/3 \rfloor$ for even n, and $p_3^s(n) \leqslant \lfloor n(n - 2)/3 \rfloor$ if n is odd. Moreover, he conjectured that this latter inequality holds for all n, $n \not\equiv 4$ (mod 6). The exact value of $p_3^s(n)$ is known only for some small values of n. (Cf., e.g., Simmons [15, 16] for the case $n = 15$, Grünbaum [5] for $n = 20$ and Harborth [8] for $n = 17$.)

As far as we know, the best lower bound by now, $p_3^s(n) > (5/16 + o(1))n^2$, was due to Füredi and J. Pach (unpublished). We are now in a position to establish a sharper lower bound.

PROPERTY 1. $p_3(\mathscr{A}_n) \geqslant n(n - 3)/3$; hence $p_3^s(n) = n^2/3 + \mathcal{O}(n)$.

PROOF. Let (L_i, L_j) denote the intersection point of lines

$$L_i = L((2i + 1)\pi/n) \quad \text{and} \quad L_j = L((2j + 1)\pi/n).$$

Using the Lemma, we obtain that only $L((2n - 2i - 2j - 2)\pi/n)$ may cross L_i and L_j at (L_i, L_j). But this line does not belong to $\mathscr{A}_n$, by definition. Therefore, the lines $L_i, L_j, L_{n-i-j-1}$ and $L_i, L_j, L_{n-i-j-2}$, resp., necessarily form triangular cells.

4. Quadrangles in a simple arrangement. Grünbaum conjectured that $p_4(\mathscr{A}) \geqslant 1$ for any simple arrangement with $n(\mathscr{A}) > 16$. (See 2.12 in [5], cf. [15, 16].) As far as we know one cannot find in the literature any example of a simple arrangement containing $\leqslant o(n^2)$ quadrangles. In view of this, Grünbaum's conjecture is surprisingly modest. It is easy to prove that

PROPERTY 2. *The simple arrangement* $\mathscr{A}_n$ *contains only* 3 *or* 5 *quadrangles for odd* n, *and* $p_4(\mathscr{A}_n) = \mathcal{O}(n)$ *is valid for all values of* n.

The results of §§3 and 4 are collected in Table 1. (We remark that $p_j(\mathscr{A}_n) = 0$ for $j \geqslant 7$.)

<div align="center">TABLE 1</div>

$n \geqslant 5$	$p_3(\mathscr{A}_n)$	$p_4(\mathscr{A}_n)$	$p_5(\mathscr{A}_n)$	$p_6(\mathscr{A}_n)$
$n \equiv 0 \pmod 6$	$\frac{1}{3}(n^2 - 3n)$	$n/2 + 6$	$n - 6$	$\frac{1}{6}(n^2 - 6n + 6)$
$n \equiv \pm 1 \pmod 6$	$\frac{1}{3}(n^2 - 3n + 5)$	5	$2n - 9$	$\frac{1}{6}(n^2 - 9n + 20)$
$n \equiv \pm 2 \pmod 6$	$\frac{1}{3}(n^2 - 3n + 8)$	$n/2$	$n - 2$	$\frac{1}{6}(n^2 - 6n + 2)$
$n \equiv 3 \pmod 6$	$\frac{1}{3}(n^2 - 3n + 9)$	3	$2n - 9$	$\frac{1}{6}(n^2 - 9n + 24)$

5. Triangles in arbitrary arrangements. Grünbaum conjectures that for any arrangement $\mathscr{A}$ of n lines, $p_3(\mathscr{A}) \leqslant n(n - 1)/3$ holds. (See [5].) Let $p_3(n) = \max\{p_3(\mathscr{A}): n(\mathscr{A}) = n\}$. The best upper bound was given by Purdy [11, 12], who proved that

$$p_3(n) \leqslant \tfrac{7}{18}n(n - 1) + \tfrac{1}{3} \quad \text{for } n \geqslant 6.$$

The best lower bound, $p_3(n) \geqslant 4 + n(n - 3)/3$, is due to Strommer [18]. His result uses a construction of Burr, Grünbaum and Sloane [1].

PROPERTY 3. $p_3(\mathscr{B}_n) \geqslant 4 + n(n - 3)/3$.

More exactly $p_3(\mathscr{B}_n) = n(n - 3)/3 + 6 - 2\varepsilon/3$, where $\varepsilon = 0, 2, 2$ according to whether $n \equiv 0, 1, 2 \pmod 3$. Further, $p_4(\mathscr{B}_n) = n - 6 + \varepsilon$, and $p_j(\mathscr{B}_n) = 0$ for $j \geqslant 5$. The proof is easy.

6. The orchard problem. Given a vertex V in an arrangement $\mathscr{A}$, denote by $t(V, \mathscr{A})$ the *multiplicity* of V, i.e. the number of lines of $\mathscr{A}$ incident to V. Further, let $t_j(\mathscr{A})$ denote the number of vertices of multiplicity j ($2 \leqslant j \leqslant n$). We use the

notation $t_j(n) = \max\{t_j(\mathscr{A}): n(\mathscr{A}) = n\}$. The "orchard problem" has been investigated for about 150 years. It can be formulated as follows: find the value of $t_3(n)$. Significant progress has been made by Burr, Grünbaum and Sloane [1]. They proved that $t_3(n) \geqslant 1 + \lfloor n(n-3)/6 \rfloor$ by construction using elliptic integrals. Moreover, they conjectured that this result is sharp if n is large enough. (Confer [1] for a complete (historical) bibliography of the subject.) Our construction (see Example 2) is much simpler, but this is only a special case of their idea. (The equation of the poles of $L(\alpha)$'s is $(x^2 + y^2)(3x - 1) = 4x^3$. This can be transformed into the form $y^2 = 4x^3 - (1/12)x - (1/216)$. Cf. [1].)

PROPERTY 4. $t_2(\mathscr{B}_n) = n - 3 + \varepsilon$, $t_3(\mathscr{B}_n) = 1 + \lfloor n(n-3)/6 \rfloor$ and $t_j(\mathscr{B}_n) = 0$ for $j \geqslant 4$.

It should be noted that recently Szemerédi and Trotter [19] proved that there exist c and c' positive real numbers such that $cn^2/k^3 < \max_{\mathscr{A}} \sum_{i \geqslant k} t_i(\mathscr{A}) < c'n^2/k^3$ for all $n > k^2$.

7. Two-coloring of arrangements.

It is easy to prove by induction that the cells of a (not necessarily simple) arrangement $\mathscr{A}$ in the *Euclidean* plane can be colored by two colors (e.g., black and white) so that any two regions with a common side get different colors. Let $b = b(\mathscr{A})$ and $w = w(\mathscr{A})$ denote the numbers of black and white polygons. Without loss of generality we can assume that $b \geqslant w$. L. Fejes Tóth [3] raised the following question: What is the maximum of the ratio b/w? Palásti [10] proved that $b/w < 2$ for $n(\mathscr{A}) \leqslant 9$. Upper bounds were given by Grünbaum [6], Simmons and Wetzel [17] and for higher dimensions by Purdy and Wetzel [13]. However, the exact value of $\max b/w$ is known only for some small values of n with $n \leqslant 16$.

Grünbaum proved that $b \leqslant 2w - 2$ for all arrangements $\mathscr{A}$ with $n(\mathscr{A}) \geqslant 3$. For $n = 3, 5, 9$ and 15 equality holds. Our Example 1 shows

PROPERTY 5. *The number of black regions* $b(\mathscr{A}_n) = (n^2 + \varepsilon)/3$, *where* $\varepsilon = 0, 2, 2$ *if* $n \equiv 0, 1, 2 \pmod 3$. *So we get* $b(\mathscr{A}_n) = 2w(\mathscr{A}_n) - (n + 2 - \varepsilon)$.

8. Gallai points.

Let $\mathscr{A}$ be an arrangement of n lines of the projective plane such that it does not contain a common point (i.e., $t_n(\mathscr{A}) = 0$). T. Gallai [4] proved that in this case there exist two lines from $\mathscr{A}$ whose intersection point has multiplicity 2. This statement was improved by Kelly and Moser [9] ($t_2(\mathscr{A}) \geqslant 3n/7$) and recently by S. Hansen [7] ($t_2(\mathscr{A}) \geqslant \lfloor n/2 \rfloor$).

The following question was posed by P. Erdős [2]. Let us suppose that the arrangement $\mathscr{A}$ does not contain a point with multiplicity more than 3. Then does there exist a Gallai triangle, i.e. three lines from $\mathscr{A}$ such that their three intersection points have multiplicity 2, or not? Our construction $\mathscr{B}_n$ shows that the answer is negative for $n \geqslant 4$, $n \not\equiv 0 \pmod 9$. Another problem on Gallai points can be found in [2].

9. Acknowledgements.

References

1. S. Burr, B. Grünbaum and N. J. A. Sloane, *The orchard problem*, Geom. Dedicata **2** (1974), 397–424.

2. P. Erdős, *Combinatorial problems in geometry*, Math. Chronical **12** (1983), 35–54.

3. L. Fejes Tóth, *A combinatorial problem concerning oriented lines in the plane*, Amer. Math. Monthly **82** (1975), 387–389.

4. T. Gallai, See H. S. M. Coxeter: *Introduction to geometry*, Wiley, New York and London, 1961.

5. B. Grünbaum, *Arrangements and spreads*, Amer. Math. Soc., Providence, R.I., 1972.

6. _____, *Two-coloring the faces of arrangements*, Period. Math. Hungar. **11** (1980), 181–185.

7. S. Hansen, *Contributions to the Sylvester-Gallai-theory*, Ph.D. Thesis, Copenhagen, 1981.

8. H. Harborth, *Two-colorings of simple arrangements*, Colloq. Math. Soc. János Bolyai, Eger, 1981 (to appear).

9. L. M. Kelly and W. O. J. Moser, *On the number of ordinary lines determined by n points*, Canad. J. Math. **10** (1958), 210–219.

10. I. Palásti, *The ratio of black and white polygons of a map generated by general straight lines*, Period. Math. Hungar. **7** (1976), 91–94.

11. G. B. Purdy, *Triangles in arrangements of lines*, Discrete Math. **25** (1979), 157–163.

12. _____, *Triangles in arrangement of lines*. II, Proc. Amer. Math. Soc. **79** (1980), 77–81.

13. G. B. Purdy and J. E. Wetzel, *Two-coloring inequalities for Euclidean arrangements in general position*, Discrete Math. **31** (1980), 53–58.

14. S. Roberts, *On the figures formed by the intercepts of a system of straight lines in a plane, and on analogous relations in space of three dimensions*, Proc. London Math. Soc. **19** (1887–1888), 405–422.

15. G. J. Simmons, *A quadrilateral-free arrangement of sixteen lines*, Proc. Amer. Math. Soc. **34** (1972), 317–318.

16. _____, *A maximal 2-arrangement of sixteen lines in the projective plane*, Period. Math. Hungar. **4** (1973), 21–23.

17. G. J. Simmons and J. E. Wetzel, *A two-coloring inequality for Euclidean two-arrangements*, Proc. Amer. Math. Soc. **77** (1979), 124–127.

18. T. O. Strommer, *Triangles in arrangements of lines*, J. Combin. Theory Ser. A **23** (1977), 314–320.

19. E. Szemerédi and W. T. Trotter, *Extremal problems in discrete geometry*, Combinatorica **3** (1983).

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