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THE CHAIN RECURRENT SET FOR MAPS OF THE CIRCLE

LOUIS BLOCK AND JOHN E. FRANKE

ABSTRACT. For a continuous map of the circle to itself we give necessary and sufficient conditions for the chain recurrent set to be precisely the set of periodic points. We also examine the possible types of examples which can occur, where the set of periodic points is closed and nonempty, but there are nonperiodic, chain recurrent points.

1. Introduction. In studying the dynamics of a map f from a space to itself, invariant sets, whose orbits satisfy some type of recurrence property, play an important role. Three such sets are the set of periodic points, the nonwandering set and the chain recurrent set, which we will denote respectively by $P(f), \Omega(f)$ and $R(f)$. In general, $P(f) \subset \Omega(f) \subset R(f)$, but equality need not hold. An obvious necessary condition for $P(f) = \Omega(f)$ or $P(f) = R(f)$ is that $P(f)$ must be closed. For maps of the interval this condition is sufficient to guarantee that $P(f) = \Omega(f) = R(f)$ [**2**, **4**, **8**, **10**, **14**]. For maps of the circle a necessary and sufficient condition for $P(f) = \Omega(f)$ is that $P(f)$ must be closed and nonempty [**1**]. In this paper we give necessary and sufficient conditions for $P(f) = R(f)$ for maps of the circle.

To give these conditions we must introduce the concept of a generalized attracting neighborhood. Let f be a continuous map of the circle to itself and let p be a periodic point of f of (least) period n. A *generalized attracting neighborhood* of p is an open neighborhood V_p of p with $\overline{V}_p \neq S^1$ and $f^n(\overline{V}_p) \subset V_p$. It is important in the definition that V_p is open and its closure is mapped inside V_p by f^n. For example, let A be any closed subset of $[0,1]$ with $0 \in A$ and $1 \in A$. There is a homeomorphism g of $[0,1]$ onto itself with $g(x) = x$ if $x \in A$ and $g(x) < x$ if $x \notin A$. Let f be the homeomorphism of the circle, S^1, onto itself obtained from g by identifying 0 and 1. The reader should verify that no fixed point of f has a generalized attracting neighborhood. Note also that in this example $P(f)$ is the nonempty closed set A, while $R(f) = S^1$.

On the other hand, let g be a homeomorphism of $[0,1]$ onto itself which has six fixed points $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and 1 with $g(x) > x$ if $x \in (0, \frac{1}{5}) \cup (\frac{2}{5}, \frac{3}{5}) \cup (\frac{3}{5}, \frac{4}{5})$ and $g(x) < x$ if $x \in (\frac{1}{5}, \frac{2}{5}) \cup (\frac{4}{5}, 1)$. The reader should verify that if f is the homeomorphism of the circle to itself obtained from g then each of the five fixed points of f has a generalized attracting neighborhood. Thus, while a sink (or contracting periodic point) always has a generalized attracting neighborhood, a source (or expanding periodic point) or a nonhyperbolic periodic point may also have a generalized attracting neighborhood.

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THEOREM A. *Let f be a continuous map of the circle to itself. $R(f) = P(f)$ if and only if $P(f)$ is closed and nonempty and for every $x \in S^1 \backslash P(f)$, some element of $\omega(x)$ has a generalized attracting neighborhood.*

In Theorem A, $\omega(x)$ denotes the set of limit points of the orbit of x. Note that if $P(f)$ is closed and nonempty, $\Omega(f) = P(f)$ [**1**], and hence any element of $\omega(x)$ is a periodic point. The proof of Theorem A shows that if $R(f) = P(f)$ and $x \in S^1 \backslash P(f)$, then every element of $\omega(x)$ has a generalized attracting neighborhood.

We remark that Theorem A is similar in flavor to results in other settings which give necessary and sufficient conditions for the nonwandering set to equal the chain recurrent set (see [**12**, Theorem 3.11; **3**, **5**, **7**, **9**, **13**]). However, one cannot extend Theorem A to show that $\Omega(f) = R(f)$ if and only if for every $x \in S^1 \backslash \Omega(f)$ some element of $\omega(x)$ has a generalized attracting neighborhood. The "if" part of this statement would imply that $\Omega(f) = R(f)$ for any map of an interval to itself (since any map of an interval may be extended to a map of the circle which is not onto), but $\Omega(f)$ need not equal $R(f)$ for maps of the interval, as an example in [**2**] shows.

The remainder of the paper is concerned with the (somewhat technical) problem of determining what types of examples are possible of maps f of the circle with $P(f)$ closed and nonempty but $P(f) \neq R(f)$. Such a map f must have a nonperiodic point x such that all elements of $\omega(x)$ are perodic but do not have generalized attracting neighborhoods. Clearly this can occur if f has a nonhyperbolic periodic point which does not have a generalized attracting neighborhood as in the first example above. Thus, we restrict our attention to the case where any nonhyperbolic periodic point has a generalized attracting neighborhood, i.e. where any periodic point is either expanding (see §2 for definition) or has a generalized attracting neighborhood. An example in this case would require a point x such that all elements of $\omega(x)$ are expanding periodic points. The next theorem shows that this is impossible if the orbit of x is infinite.

THEOREM B. *Suppose that the orbit of a point x (under a continuous map of the circle to itself) is infinite. Then some element of $\omega(x)$ is not an expanding periodic point.*

One can construct examples where the orbit of x is finite and all elements of $\omega(x)$ are expanding periodic points (i.e. x is eventually an expanding periodic point) which do not have generalized attracting neighborhoods. For example, let g be the map of $[0,1]$ to itself with $g(0) = 0$, $g(\frac{1}{4}) = 1$, $g(\frac{1}{2}) = \frac{9}{16}$, $g(\frac{3}{4}) = \frac{11}{16}$ and $g(1) = 1$, such that g is linear on each of the intervals $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$. Let f be the map of the circle to itself obtained from g. Then $\Omega(f)$ consists of the expanding fixed point, 0, and the contracting fixed point, $\frac{5}{8}$, but the point $\frac{1}{4}$ is chain recurrent as are its inverse images.

The previous example yields a continuous map f of the circle to itself with $P(f)$ closed and nonempty such that any periodic point is either expanding or has a generalized attracting neighborhood, but $R(f) \neq P(f)$. Note that in the example there is a critical point x (see §2 for definition) such that for some positive integer k, $f^k(x)$ is an expanding periodic point which has no generalized attracting neighborhood (i.e. $x = \frac{1}{4}$ and $k = 1$). Our final theorem shows that this must occur in any such example.

THEOREM C. *Let f be a continuous map of the circle to itself. Suppose that $P(f)$ is closed and nonempty and that any periodic point of f is either expanding or has a generalized attracting neighborhood. Suppose also that for any critical point x of f, if $f^k(x)$ is periodic for some positive integer k, then $f^k(x)$ has a generalized attracting neighborhood. Then $R(f) = P(f)$.*

2. Background. Let f be a continuous map from a compact metric space (X, d) into itself. An *ε-chain* from x to y (where $x, y \in X$) is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X with $x = x_0$, $y = x_n$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. We say *x can be chained to y* if for every $\varepsilon > 0$ there is an ε-chain from x to y, and we say x is *chain recurrent* if x can be chained to x. The set of all chain recurrent points is denoted by $R(f)$.

A subset Y of X is called *positively chain invariant* if for every $y \in Y$ and $x \in X \backslash Y$, y cannot be chained to x. Proofs of the following two lemmas may be found in [**2**]. The first lemma was also noted in [**11**].

LEMMA 1. *$R(f) = R(f^n)$ for any positive integer n.*

LEMMA 2. *Let Y be a positively chain invariant subset of X. If $x \notin Y$ and $f^k(x) \in Y$ for some positive integer k, then $x \notin R(f)$.*

If $Y \subset X$ we let $f \mid Y$ denote the restriction of f to Y and $\overline{Y}$ denote the closure of Y.

LEMMA 3. *If Y is an open subset of X such that $f(\overline{Y}) \subset Y$ then $\overline{Y}$ is positively chain invariant and $R(f) \cap \overline{Y} = R(f \mid \overline{Y})$.*

PROOF. Let $\varepsilon_1 = d(X \backslash Y, f(\overline{Y}))$. Note that $\varepsilon_1 > 0$ since $X \backslash Y$ and $f(\overline{Y})$ are disjoint compact sets. If $y \in \overline{Y}$ and $0 < \varepsilon < \varepsilon_1$, then every ε-chain starting at y is completely contained in $\overline{Y}$. Thus $\overline{Y}$ is positively chain invariant and $R(f) \cap \overline{Y} = R(f \mid \overline{Y})$. Q.E.D.

The following lemma, which slightly relaxes the hypothesis of the previous lemma in the case where Y is an open interval, is proved as in Lemma 4 of [**2**]. The lemma allows for the case where one endpoint of Y is mapped to the other.

LEMMA 4. *Let f be a continuous map of S^1 to itself. Let Y be a proper open subinterval of S^1 with $f(\overline{Y}) \subset \overline{Y}$. If a and b are the endpoints of Y and $f(a) \neq a$ while $f(b) \in Y$, then $\overline{Y}$ is positively chain invariant.*

For the rest of the paper, f will denote a continuous map of the circle to itself and d will denote a metric on the circle. A point $x \in S^1$ is called a *critical point* of f if f fails to be injective on every neighborhood of x.

Let p be a periodic point of f of (least) period n which is not a critical point of f^n. Let $k = n$ if f^n preserves orientation at p, and $k = 2n$ if f^n reverses orientation at p. We call p an *expanding* periodic point if there is an open neighborhood V_p of p such that for each $x \in V_p$, $d(f^k(x), p) > d(x, p)$.

If p is a fixed point of f, its unstable manifold is defined to be

$$W^u(p, f) = \bigcap \left(\bigcup_{n=0}^{\infty} f^n(V) \right),$$

where the intersection is taken over all neighborhoods V of p. If p is an expanding fixed point then $W^u(p,f)$ is an invariant interval with p as an interior point. We will use the following lemma in the proof of Theorem C.

LEMMA 5. *Let f be a continuous map of the circle to itself and let p be an expanding fixed point of f. Let q be an endpoint of $W^u(p,f)$.*

(1) *If $q \in W^u(p,f)$, then q is in the orbit of a critical point of f.*

(2) *If $q \notin W^u(p,f)$, then q is not expanding and is either a fixed point of f or a point of period two.*

PROOF. There is an open neighborhood V of p with $W^u(p,f) = \bigcup_{n=0}^{\infty} f^n(V)$.

If $q \in W^u(p,f)$ there is a point $r \in V$ and a positive integer n such that $f^n(r) = q$. One of the points $r, f(r), \ldots, f^{n-1}(r)$ must be a critical point, or else $f^r(V)$ would contain a neighborhood of q and q would not be an endpoint of $W^u(p,f)$. This proves (1).

Now suppose that $q \notin W^u(p,f)$. Since

$$f(W^u(p,f)) = W^u(p,f) \quad \text{and}$$
$$f(\overline{W^u(p,f)}) = \overline{W^u(p,f)},$$

q is either a fixed point of f or a point of period two. In the latter case, we may replace f by f^2, so we assume that q is a fixed point of f.

Let $\overline{W^u(p,f)} = [q_1, q]$. Note that $f(q_1) \neq q$ (because if $f(q_1) = q$ then $q_1 \in W^u(p,f)$ and hence $q \in W^u(p,f)$).

Suppose q is an expanding fixed point of f. Since q is an endpoint of $W^u(p,f)$, f preserves orientation at q. Let $V_q = (a,b)$ be a neighborhood of q such that $d(f(x),q) > d(x,q)$ for all $x \in V_q$, and $f(x) \in (p,q)$ for all $x \in (a,q)$. Then $f([q_1,a]) = [q_1,c]$, where $c \in (a,q)$, and $f([q_1,c]) \subset [q_1,c]$. This implies that $q \notin \overline{W^u(p,f)}$, a contradiction. This proves (2). Q.E.D.

We conclude this section with a lemma which shows that if one point in a periodic orbit has a generalized attracting neighborhood then the others do also.

LEMMA 6. *Let y be a periodic point of a continuous map f of the circle to itself. If y has a generalized attracting neighborhood then so does each point in the orbit of y.*

PROOF. Let n denote the period of y and let $y_n = f^{n-1}(y)$. It suffices to show that y_n has a generalized attracting neighborhood.

Let V_1 be a connected generalized attracting neighborhood of y. Let V_n denote the component of $f^{-1}(V_1)$, which contains y_n. Then V_n is an open interval and $f(\overline{V}_n) \subset \overline{V}_1$.

If $\overline{V}_n = S^1$, then f is not onto and any periodic point has a generalized attracting neighborhood. Thus, we may assume that $\overline{V}_n \neq S^1$. It follows that $\overline{V}_n$ is a proper closed interval $[a,b]$ on S^1 with $f(a) \notin V_1$ and $f(b) \notin V_1$. Hence, $a \notin f^{n-1}(\overline{V}_1)$ and $b \notin f^{n-1}(\overline{V}_1)$. Thus,

$$f^n(\overline{V}_n) = f^{n-1}(f(\overline{V}_n)) \subset f^{n-1}(\overline{V}_1) \subset V_n,$$

so V_n is a generalized attracting neighborhood of y_n. Q.E.D.

3. Proofs of the theorems.

PROOF OF THEOREM A. First suppose $R(f) = P(f)$. Since $R(f)$ is closed and nonempty, so is $P(f)$. Let $x \in S^1 \backslash P(f)$ and let $y \in \omega(x)$. Then $y \in \Omega(f) = P(f)$. We will show that y has a generalized attracting neighborhood.

Let $R_\varepsilon(y)$ denote the set of $z \in S^1$ such that there is an ε-chain from y to z. It follows from the definition of ε-chain that $R_\varepsilon(y)$ is open. Since $x \notin R(f)$, for some $\varepsilon > 0$, $x \notin R_\varepsilon(y)$.

Let V_y denote the component of $R_\varepsilon(y)$ that contains y. Then V_y is an open interval, invariant under f^n, where n is the period of y. In fact, since there is no ε-chain from y to the endpoints of V_y, for any $z \in V_y$ the distance from $f^n(z)$ to any endpoint of V_y is at least ε. Thus, $f^n(\overline{V}_y) \subset V_y$. If f is onto then $\overline{V}_y \neq S^1$ and V_y is a generalized attracting neighborhood of y, while if f is not onto it follows easily that any periodic point of f (and in particular y) has a generalized attracting neighborhood.

Now suppose that $P(f)$ is closed and nonempty and for every $x \in S^1 \backslash P(f)$, some element of $\omega(x)$ has a generalized attracting neighborhood. Let $x \in S^1 \backslash P(f)$. We will show that $x \notin R(f)$.

Let y be an element of $\omega(x)$ which has a generalized attracting neighborhood and denote the orbit of y by $\{y_1, y_2, \ldots, y_n\}$. By Lemma 6, each y_k has a generalized attracting neighborhood V_k. Since $f^n \mid \overline{V}_k$ is a map of a compact interval to itself, it follows from [**2**] that $R(f^n \mid \overline{V}_k) = P(f^n \mid \overline{V}_k)$. Thus, if $x \in \overline{V}_k$ for some k then $x \notin R(f^n \mid \overline{V}_k)$. Hence, by Lemmas 1 and 3, $x \notin R(f)$.

Thus, we may assume that $x \notin \overline{V}_k$ for each $k = 1, \ldots, n$. Since $y \in \omega(x)$, for some $k = 1, \ldots, n$ and some positive integer j, $f^{jn}(x) \in V_k$. By Lemmas 2 and 3, $x \notin R(f^n)$, so by Lemma 1, $x \notin R(f)$.

PROOF OF THEOREM B. Suppose each element of $\omega(x)$ is an expanding periodic point. Let $y_1 \in \omega(x)$. Let $n_1 = 2j_1$, where j_1 is the period of y_1. Let ε_1 denote the distance from y_1 to the set of periodic points z of f such that $z \neq y_1$ and the period of z is at most n_1. Since y_1 is an expanding point, $\varepsilon_1 > 0$. There are neighborhoods W_1 and V_1 of y_1 such that f^{n_1} is expanding on V_1, $f^{n_1}(W_1) \subset V_1$, and the diameter of V_1 is less than $\varepsilon_1/3$. It follows that there is a point y_2 in $\omega(x) \cap (\overline{V}_1 \backslash \overline{W}_1)$.

Let O_1 be the union of neighborhoods of diameter $\varepsilon_1/3$ about each periodic point $z \neq y_1$ whose period is at most n_1 and let $K_1 = S^1 \backslash (O_1 \cup W_1)$. Then y_2 is in the interior of K_1. Let $n_2 = 2j_2$, where j_2 is the period of y_2, and note that $n_2 > n_1$. Let ε_2 denote the distance from y_2 to the set of periodic points z of f such that $z \neq y_2$ and the period of z is at most n_2. Let O_2 be the union of neighborhoods of diameter $\varepsilon_2/3$ about each periodic point $z \neq y_2$ whose period k satisfies $n_1 < k \leq n_2$. Let W_2 and V_2 be neighborhoods of y_2 with $\overline{V}_2$ contained in the interior of K_1 such that f^{n_2} is expanding on V_2, $f^{n_2}(W_2) \subset V_2$, and the diameter of V_2 is less than $\varepsilon_2/3$. There is a point y_3 in $\omega(x) \cap (\overline{V}_2 \backslash \overline{W}_2)$. Let

$$K_2 = S^1 \backslash (O_1 \cup W_1 \cup O_2 \cup W_2)$$

and note that y_3 is in the interior of K_2.

Define K_n inductively as above, and let $G_n = K_n \cap \omega(x)$. Then $\{G_n\}$ is a decreasing family of nonempty compact sets, so $\bigcap_{n=1}^{\infty} G_n$ is nonempty. Any point in the intersection is in $\omega(x)$ but is not periodic, a contradiction.

PROOF OF THEOREM C. Let $x \in S^1$ and suppose that x is not periodic. We must show that x is not chain recurrent. We remark that if any element q of $\omega(x)$ has a generalized attracting neighborhood V_q, then x is not chain recurrent. This follows as in the proof of Theorem A. Thus, we may assume that each element of $\omega(x)$ is an expanding periodic point. By Theorem B, x is eventually periodic.

By the remark above and Lemma 1, we may assume that for some positive integer k, $f^k(x)$ is an expanding fixed point which has no generalized attracting neighborhood. Then, by hypothesis, none of the points $x, f(x), \ldots, f^{k-1}(x)$ are critical points. This implies that f^k maps an open interval about x onto an open interval about $f^k(x)$. Thus, if $x \in \overline{W^u(f^k(x), f)}$, then $x \in \Omega(f) = P(f)$. Hence $x \notin \overline{W^u(f^k(x), f)}$.

Let q_1 and q_2 denote the (distinct) endpoints of $\overline{W^u(f^k(x), f)}$. We consider three cases.

First, suppose that neither q_1 nor q_2 is a fixed point of f^2. Then, by Lemma 4, $\overline{W^u(f^k(x), f)}$ is positively chain invariant. Since this set contains $f^k(x)$ but not x, x is not chain recurrent by Lemma 2.

Second, suppose that q_1 and q_2 are both fixed points of f^2. Replacing f by f^2 if necessary, we may assume that q_1 and q_2 are both fixed points of f. By Lemma 5 and our hypothesis, q_1 and q_2 have generalized attracting neighborhoods V_{q_1} and V_{q_2}. If $x \in \overline{V}_{q_1}$ or $x \in \overline{V}_{q_2}$, then x is not chain recurrent by Lemma 3 and the theorem of [**2**]. hence, we may assume that $x \notin (\overline{V}_{q_1} \cup \overline{V}_{q_2})$. It follows from Lemma 3 that $\overline{W^u(f^k(x), f) \cup V_{q_1} \cup V_{q_2}}$ is positively chain invariant. Since this set contains $f^k(x)$ but not x, x is not chain recurrent by Lemma 2.

Finally, suppose that $f^2(q_1) = q_1$ and $f^2(q_2) \neq q_2$. If $f(q_1) \neq q_1$, it follows as in the first case that x is not chain recurrent. If $f(q_1) = q_1$, then as in the second case q_1 has a generalized attracting neighborhood V_{q_1} and x is not chain recurrent.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695-8205