PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 4, December 1984

JET-DETECTABLE EXTREMA

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ABSTRACT. Given a real polynomial f with $f(0) = 0$, we define a finite family of germs of smooth one-variable functions that can be used to test if f has an extreme value at 0. A generalization is obtained for functions that have k-jets in a suitable sense.

0. Introduction. We consider germs $f\colon (\mathbf{R}^n;0) \to (\mathbf{R};0)$. As usual, we also denote by f any representative of the germ. Our aim is to indicate conditions on the punctual jet of f at $0 \in \mathbf{R}^n$ in order to decide if this point is a local extremum of f.

The idea is to reduce the study to the local behavior at $0 \in \mathbf{R}^+$ of finitely many germs $f_i\colon (\mathbf{R}^+;0) \to (\mathbf{R};0)$ of functions of a single variable. We start with the case in which f is a polynomial and give a complete characterization. Then we go to the case in which f has "generalized k-jet"; the characterization now is no longer complete, but is the best one in the sense that all germs are classified except those whose k-jets do not contain the desired information.

1. Extremum of polynomials.

1.0 DEFINITION. Let $f\colon (\mathbf{R}^n;0) \to (\mathbf{R};0)$ be a polynomial of degree $\leq k$ and $g(x) = |x|^2$, $x \in \mathbf{R}^n$. We denote by $V(f)$ the set of those $x \in \mathbf{R}^n$, $x \neq 0$, where $\operatorname{grad} f$ is radial. It is easy to see that

$$V(f) = \left\{ x \in \mathbf{R}^n | x \neq 0 \text{ and } \operatorname{rank} \begin{vmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{vmatrix} < 2 \right\}.$$

Let $S_\varepsilon = \{x \in \mathbf{R}^n \mid |x| = \varepsilon\}$ and $B_\varepsilon = \{x \in \mathbf{R}^n \mid |x| < \varepsilon\}$. Then $V(f) \cap S_\varepsilon$ is precisely the set of the critical points of $f|S_\varepsilon$. In particular, for any $\varepsilon > 0$, $V(f) \cap S_\varepsilon \neq \emptyset$, which shows that $V(f)$ is a semialgebraic variety of $\mathbf{R}^n$ adherent to the origin.

We need

1.1. SARD'S ALGEBRAIC THEOREM. *Let V be a semialgebraic variety of $\mathbf{R}^n$ and $f\colon \mathbf{R}^n \to \mathbf{R}$ a polynomial. Then the set of critical values of $f|V$ is finite.*

PROOF. V admits a stratification with finitely many strata, each one semialgebraic [**G**]. It is enough to consider the case where V is a semialgebraic submanifold of $\mathbf{R}^n$.

The set $A = \{x \in V |\ x \text{ is a critical point of } f|V\}$ is semialgebraic by the Tarski-Seidenberg theorem [**M**], since $A = V \setminus \pi\{TV \setminus \ker Df\}$, where $\pi\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$

Received by the editors September 21, 1983.
1980 *Mathematics Subject Classification*. Primary 57R45, 58E15.
[1]Partially supported by the Conselho Nacional para o Desenvolvimento Cientifíco e Tecnológico, Brazil and by the Fundação de Amparo à Pesquisa do Estado de São Paulo, Brazil.

is the first projection, and TV and $\ker Df$ are semialgebraic in $\mathbf{R}^n \times \mathbf{R}^n$. Therefore, A has finitely many connected components [**G**]. Since $f|A$ is locally constant, it is constant in each connected component. □

Assume that $f\colon (\mathbf{R}^n; 0) \to (\mathbf{R}; 0)$ is a polynomial. Hence $V(f)$ admits a canonical Whitney stratification in finitely many connected strata, each of which is a semialgebraic submanifold of $\mathbf{R}^n$. By the above theorem there exists an $\varepsilon_0 > 0$ such that $\forall \varepsilon$, $0 < \varepsilon < \varepsilon_0 \Rightarrow S_\varepsilon \pitchfork V_i$, $\forall i$. This proves

1.2. PROPOSITION. *For $0 < \varepsilon < \varepsilon_0$, $V(f) \cap B_\varepsilon$ has finitely many connected components K_i, each one semialgebraic and adherent to the origin.*

We proceed to define the germs $\phi_i\colon (\mathbf{R}^+; 0) \to (\mathbf{R}; 0)$ whose behavior will detect the existence of an extremum of f.

Corresponding to each K_i, let $\phi_i\colon (\mathbf{R}^+; 0) \to (\mathbf{R}; 0)$ be defined by $\phi_i(0) = 0$ and, for $0 < \varepsilon < \varepsilon_0$, $\phi_i(\varepsilon) = f(x)$, where $x \in K_i \cap S_\varepsilon$. ϕ_i is well defined since $f|K_i \cap S_\varepsilon$ is constant, because $K_i \cap S_\varepsilon$ is connected and $\operatorname{grad}(f|K_i \cap S_\varepsilon) = 0$.

We have thus associated to the polynomial f a finite number of germs ϕ_i of a single variable.

1.3. THEOREM. (a) *A necessary and sufficient condition for f to admit a minimum (a strong minimum, a maximum, a strong maximum) at $0 \in \mathbf{R}^n$ is that the same holds true for each ϕ_i at $0 \in \mathbf{R}^+$.*

(b) *A necessary and sufficient condition for f not to admit an extremum at $0 \in \mathbf{R}^n$ is that there exist i, j such that ϕ_i has a strong maximum and ϕ_j a strong minimum at $0 \in \mathbf{R}^+$.*

PROOF. We prove only that the condition of (a) is sufficient; the rest follows easily. Suppose, for example, that all ϕ_i have a minimum at $0 \in \mathbf{R}^+$. Since we have a finite number of ϕ_i, there exists an $\varepsilon_1 > 0$ ($\varepsilon_1 \leq \varepsilon_0$) such that $\forall \varepsilon > 0$, $\varepsilon < \varepsilon_1$, $\forall_i\, \phi_i(\varepsilon) \geq 0$. The set of the critical points of $f|S_\varepsilon$ is $V(f) \cap S_\varepsilon$. Therefore the minimum value of $f|S_\varepsilon$ is $\phi_i(\varepsilon)$ for some i, which shows that $f|B_{\varepsilon_1} \geq 0$, so f has a minimum at $0 \in \mathbf{R}^n$. The other cases may be treated similarly. □

1.4. REMARK. K_i is semialgebraic and $0 \in \overline{K_i}$. Hence, by the curve selection lemma [**M**], there exists an analytic curve $\gamma_i\colon [0, 1) \to \mathbf{R}^n$ such that $\gamma_i(0) = 0$ and $\gamma_i(t) \in K_i$ for $t \neq 0$. Note that

$$\gamma_i(t) = a_{m_i} t^{m_i} + \sum_{k=m_i+1}^{\infty} a_k t^k$$

with $a_{m_i} \neq 0$, $m_i \geq 1$ and $a_k \in \mathbf{R}^n$. Thus $h_i = |\gamma_i|$ is a germ of homeomorphism at $0 \in \mathbf{R}^+$ because h_i is continuous and locally strictly increasing. Since $f \circ \gamma_i = \phi_i \circ h_i$ and $f \circ \gamma_i$ is analytic, we have $\phi_i \equiv 0$, or 0 is an isolated zero of ϕ_i, and so a strong extremum. Moreover, since h_i is increasing, if ϕ_i has a minimum (maximum), so has $f \circ \gamma_i$, and conversely. This fact can be detected by the sign of the least nonzero derivative of $f \circ \gamma_i$ at 0. Furthermore, we have

$$h_i(t) = |a_{m_i}| t^{m_i} (1 + \psi(t))^{1/2}$$

for some analytic function $\psi(t)$ with $\psi(0) = 0$. Then

$$(|a_{m_i}|/2) t^{m_i} \leq h_i(t) \leq 2 |a_{m_i}| t^{m_i},$$

or, equivalently,

$$(|a_{m_i}|/2)[h_i^{-1}(u)]^{m_i} \le u \le 2|a_{m_i}|[h_i^{-1}(u)]^{m_i}.$$

These inequalities will be used in §3.

2. Generalized jets.

2.0. DEFINITION. We say that the germ $f\colon (\mathbf{R}^n;0) \to (\mathbf{R};0)$ has *punctual k-jet* at $0 \in \mathbf{R}^n$ if there is polynomial $P\colon \mathbf{R}^n \to \mathbf{R}$ of degree less than k ($\le k$, $\operatorname{dg} 0 = -\infty$) such that $\lim_{x\to 0}((f(x) - P(x))/|x|^k) = 0$.

Obviously such a polynomial, if it exists, is unique and is the Taylor polynomial of f when f has one. We denote it by $j^k f$ and call it the k-jet of f.

The set of all germs $f\colon (\mathbf{R}^n;0) \to (\mathbf{R};0)$ which have a k-jet will be denoted by $\mathcal{G}^k(n)$.

Note that $\mathcal{C}^k(n) \subset \mathcal{G}^k(n) \subset \mathcal{G}^{k-1}(n)$. These inclusions are trivial and strict since the function

$$f(x) = \begin{cases} x^2, & x \in \mathbf{Q}, \\ x^2 \exp(-x^{-2}), & x \in \mathbf{R}\backslash\mathbf{Q}, \end{cases}$$

is discontinuous except at 0 and belongs to $\mathcal{G}^k(1)$ for any k.

2.1. REMARKS. The concept of k-jet has a natural extension to germs $f\colon (\mathbf{R}^n;0) \to (\mathbf{R}^p;0)$ and, in this context, the general theorems of the usual differential calculus, such as the chain-rule, the local forms of immersions and submersions, Morse's lemma, etc.

As illustrations we present the local form of submersions, when $p = 1$, and the splitting lemma.

2.2. LOCAL FORM OF SUBMERSIONS. *Let $f \in \mathcal{G}^k(n)$, $k \ge 1$. If $j^1 f \ne 0$, then for all r and s ($1 \le r \le k$; $r \le s \le \omega$) there exists a germ of diffeomorphisms $\phi\colon (\mathbf{R}^n;0) \to (\mathbf{R}^n;0)$ of class $\mathcal{C}^s$ such that $j^r(f \circ \phi) = \pi_1$.*

PROOF. It is enough to consider the case $r = k$, $s = \omega$. Since $j^k f$ is a polynomial, $j^1(j^k f) = j^1 f \ne 0$, and therefore there exists $\phi \in \operatorname{Diff}^\omega(\mathbf{R}^n;0)$ such that $(j^k f) \circ \phi = \pi_1$. Hence

$$j^k(f \circ \phi) = j^k((j^k f) \circ (j^k \phi)) = j^k((j^k f) \circ \phi) = j^k \pi_1 = \pi_1. \quad \square$$

2.3. SPLITTING LEMMA. *Let $f \in \mathcal{G}^k(n)$, $k \ge 2$. If $j^1 f = 0$ then there exist a splitting $\mathbf{R}^n = \mathbf{R}^u \times \mathbf{R}^v \times \mathbf{R}^w$ and, for all r and s ($2 \le r \le k$; $r \le s \le \infty$), a germ of diffeomorphism $\phi\colon (\mathbf{R}^n;0) \to (\mathbf{R}^n;0)$ of class $\mathcal{C}^s$ such that $j^r(f \circ \phi)(x;y;z) = |x|^2 - |y|^2 + p_r(z)$, where $p_r(z)$ is a polynomial of degree $\le r$ satisfying $j^2 p_r = 0$.*

PROOF. We consider only the case $r = k$, $s = \infty$. Using the fact that $j^1(j^k f) = 0$ and the classical splitting lemma for the polynomial $j^k f$, we can find a splitting $\mathbf{R}^n = \mathbf{R}^u \times \mathbf{R}^v \times \mathbf{R}^w$ and a $\phi \in \operatorname{Diff}^\infty(\mathbf{R}^n;0)$ such that

$$((j^k f) \circ \phi)(x;y;z) = |x|^2 - |y|^2 + g(z),$$

where $j^2 g = 0$. But $j^k(f \circ \phi) = j^k((j^k f) \circ \phi)$, which implies

$$j^k(f \circ \phi)(x;y;z) = |x|^2 - |y|^2 + j^k g(z). \quad \square$$

3. Jet-detectable extrema. Now we extend Theorem 1.3, with the obvious adaptations, to functions which have generalized k-jets.

3.0. DEFINITION. For $f \in \mathcal{G}^r(n)$ we say that f is *k-decidable* ($k \leq r$) if it is possible, using only $j^k f$, to decide if the origin is a maximum, a minimum, or not an extremum of f.

Note that if f is k-decidable and $k \leq s \leq r$, then f is s-decidable.

Let $f \in \mathcal{G}^r(n)$. We fix a $k \leq r$ and apply to the polynomial $j^k f$ the process described before. We get a finite family of germs $\phi_i \colon (\mathbf{R}^+; 0) \to (\mathbf{R}; 0)$.

We now proceed to define the germ $f_i \colon (\mathbf{R}^+; 0) \to (\mathbf{R}; 0)$ whose behavior at $0 \in \mathbf{R}^+$ will possibly allow us to decide about the existence of an extremum of f at $0 \in \mathbf{R}^n$.

With the notation of 1.4 we have $\phi_i \circ h_i = (j^k f) \circ \gamma_i$, which is analytic, and

$$\gamma_i = a_{m_i} t^{m_i} + \sum_{j=m_i+1}^{\infty} a_j t^j,$$

where $a_{m_i} \neq 0$. We put $f_i = j^{m_i k}(\phi_i \circ h_i)$.

The germ f_i so defined is the natural substitute for the k-jet of ϕ_i when ϕ_i does not have such a jet. We remark that the theorem below holds if we replace f_i by $j^k \phi_i$ when this latter exists.

3.1. THEOREM. (a) *If all f_i have a strong minimum (maximum) at $0 \in \mathbf{R}^+$, so does f at $0 \in \mathbf{R}^n$.*

(b) *If among the f_i there is one which has a strong minimum and another which has a strong maximum at $0 \in \mathbf{R}^+$, then f does not have an extremum at $0 \in \mathbf{R}^n$.*

(c) *In the remaining cases, f is not k-decidable.*

PROOF. (a) We consider only the minimum case. Since f_i is a polynomial of degree $\leq m_i k$ we can choose $\alpha_i > 0$ and $\tau_i > 0$ such that $\forall t$,

$$0 < t < \tau_i \Rightarrow f_i(t) > \alpha_i t^{m_i k} > \beta_i [h_i(t)]^k,$$

where $\beta_i > 0$ is obtained by using inequalities 1.4. Since the family is finite we may take $\beta > 0$ and $\tau > 0$ such that $\forall i$, $\forall t$, $0 < t < \tau \Rightarrow f_i(t) > \beta[h(t)]^k$.

We also have $\forall \lambda > 0$, $\exists \tau > 0$ such that $\forall i$ $\tau \geq h_i(\tau)$ and $\forall u$

$$0 < u < \tau \Rightarrow |\phi_i(u) - f_i[h_i^{-1}(u)] < \lambda [h_i^{-1}(u)]^{m_i k} < \beta u^k / 2$$

for λ small enough.

Finally, we choose $\varepsilon > 0$, $\varepsilon < \varsigma$ $\forall x$, $|x| = \delta < \varepsilon \Rightarrow |f(x) j^k f(x)| < \beta\delta^k/4$. So, we have $\forall x$,

$$|x| = \delta < \varepsilon \Rightarrow f(x) \geq \min j^k f | S_\delta - \beta\delta^k/4.$$

By Theorem 1.3 $\exists i \mid \min j^k f | S_\delta = \phi_i(\delta)$, therefore $\forall x$

$$|x| = \delta < \varepsilon \Rightarrow f(x) \geq \phi_i(\delta) - \beta|\delta|^k/4 \geq \beta\delta^k/4,$$

which shows (a).

(b) We take α, $\tau > 0$ such that $\forall t$

$$0 < t < \tau \Rightarrow f_i(t) > \alpha t^{m_i k} > \beta_i [h_i(t)]^k$$

and

$$f_j(t) < -\alpha t^{m_j k} < -\beta_j [h_j(t)]^k.$$

Let $\beta = \min(\beta_i, \beta_j) > 0$ and $\varsigma > 0$, $\varsigma < h_i(\tau)$ and $\varsigma < h_j(\tau)$ such that $\forall u$,

$$0 < u < \varsigma \Rightarrow |\phi_r(u) - f_r(h_r^{-1}(u))| < \beta u^k/2, \qquad r = i, j.$$

Finally we take $\varepsilon > 0$, $\varepsilon < \varsigma$ such that $\forall x$,

$$|x| < \varepsilon \Rightarrow |f(x) - j^k f(x)| < \beta |x|^k/4.$$

We claim that for any δ, $0 < \delta < \varepsilon$, $f|S_\delta \cap K_i > 0$ and $f|S_\delta \cap K_j < 0$. For, if $x_j \in S_\delta \cap K_j$,

$$f(x_j) \leq \phi(|x_j|) + \frac{\beta}{4}|x_j|^k \leq f_j[h_j^{-1}(|x_j|)] + \left(\frac{\beta}{2} + \frac{\beta}{4}\right)|x|^k \leq -\frac{\beta}{4}|x_j|^k < 0.$$

Similarly, if $x_i \in S_\delta \cap K_i$,

$$f(x_i) \geq \beta |x_i|^k/4 > 0,$$

which proves (b).

(c) Suppose, for example, that all f_i have a minimum but f_j has a weak minimum at $0 \in \mathbf{R}^+$ (hence $f_j \equiv 0$). It is enough to construct two germs, g_0 and g_1, having the same k-jet as f such that $0 \in \mathbf{R}^n$ is a strong minimum for g_0 and is not a minimum (not even a weak one) for g_1. To do that, for each j, such that $f_j \equiv 0$, we choose $\varepsilon_j > 0$ (small enough) and $\alpha_j > 0$ (big enough) in order that $\forall t$,

$$0 < t < \varepsilon_j \Rightarrow |\phi_j[h_j(t)]| < \alpha_j t^{m_j k + 1} < \beta_j [h_j(t)]^{k + (1/m_j)}$$

(β_j as in 1.4). Since there are finitely many such j's we take $\varepsilon, \beta > 0$, $m \in N^*$ such that for all those j's we have $\forall t$,

$$0 < t < \varepsilon \Rightarrow |\phi_j(h_j(t))| < \beta [h_j(t)]^{k + (1/m)}$$

and put

$$g_\nu(x) = j^k f(x) - (-1)^\nu \beta |x|^{k + 1/m}, \qquad \nu = 0, 1.$$

It is clear that $j^k g_\nu = j^k f$ and the critical points of $g_\nu | S_\delta$ are precisely those of $j^k f | S_\delta$, $\forall \delta > 0$. Moreover, $\forall i$,

$$g_\nu(\gamma_i(t)) = \phi_i(h_i(t)) - (-1)^\nu \beta [h_i(t)]^{k + 1/m}.$$

Let us show that $0 \in \mathbf{R}^n$ is not even a weak minimum for g_0. For if $f_j \equiv 0$ and $x_i \in K_j$ with $|x_j| < h_j(\varepsilon)$, then

$$g_0(x_j) = g_0(\gamma_j(t)) = \phi_j(h_j(t)) - \beta [h_j(t)]^{k + 1/m} < 0.$$

Let us show finally that $0 \in \mathbf{R}^n$ is a strong minimum for g_1. Let $\delta > 0$ such that $\forall j$, $f_j \equiv 0 \Rightarrow \delta < h_j(\varepsilon)$. We have $\min g_1 | S_\delta = g_1(\gamma_i(t))$ for some i, and then

$$\min g_1 | S_\delta = \phi_i(h_i(t)) + \beta [h_i(t)]^{k + 1/m}.$$

If $f_i \equiv 0$ then $\beta [h_i(t)]^{k + 1/m} > 0$, otherwise $\phi_i(h_i(t)) > \mu_i t^k$ for μ_i small enough and any t nearby 0, and again $g_1(\gamma_i(t)) > 0$. □

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