JET-DETECTABLE EXTREMA
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ABSTRACT. Given a real polynomial \( f \) with \( f(0) = 0 \), we define a finite family of germs of smooth one-variable functions that can be used to test if \( f \) has an extreme value at 0. A generalization is obtained for functions that have \( k \)-jets in a suitable sense.

0. Introduction. We consider germs \( f: (\mathbb{R}^n;0) \rightarrow (\mathbb{R};0) \). As usual, we also denote by \( f \) any representative of the germ. Our aim is to indicate conditions on the punctual jet of \( f \) at \( 0 \in \mathbb{R}^n \) in order to decide if this point is a local extremum off.

The idea is to reduce the study to the local behavior at \( 0 \in \mathbb{R}^+ \) of finitely many germs \( f_i: (\mathbb{R}^+;0) \rightarrow (\mathbb{R};0) \) of functions of a single variable. We start with the case in which \( f \) is a polynomial and give a complete characterization. Then we go to the case in which \( f \) has "generalized \( k \)-jet"; the characterization now is no longer complete, but is the best one in the sense that all germs are classified except those whose \( k \)-jets do not contain the desired information.

1. Extremum of polynomials.

1.0 Definition. Let \( f: (\mathbb{R}^n;0) \rightarrow (\mathbb{R};0) \) be a polynomial of degree \( \leq k \) and \( g(x) = |x|^2, x \in \mathbb{R}^n \). We denote by \( V(f) \) the set of those \( x \in \mathbb{R}^n, x \neq 0 \), where \( \text{grad} \ f \) is radial. It is easy to see that

\[
V(f) = \left\{ x \in \mathbb{R}^n | x \neq 0 \text{ and } \frac{\text{grad} \ f(x)}{\text{grad} \ g(x)} < 2 \right\}.
\]

Let \( S_\varepsilon = \{ x \in \mathbb{R}^n | |x| = \varepsilon \} \) and \( B_\varepsilon = \{ x \in \mathbb{R}^n | |x| < \varepsilon \} \). Then \( V(f) \cap S_\varepsilon \) is precisely the set of the critical points of \( f|S_\varepsilon \). In particular, for any \( \varepsilon > 0 \), \( V(f) \cap S_\varepsilon \neq \emptyset \), which shows that \( V(f) \) is a semialgebraic variety of \( \mathbb{R}^n \) adherent to the origin.

We need

1.1. SARD'S ALGEBRAIC THEOREM. Let \( V \) be a semialgebraic variety of \( \mathbb{R}^n \) and \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) a polynomial. Then the set of critical values of \( f|V \) is finite.

Proof. \( V \) admits a stratification with finitely many strata, each one semialgebraic \([G]\). It is enough to consider the case where \( V \) is a semialgebraic submanifold of \( \mathbb{R}^n \).

The set \( A = \{ x \in V | x \text{ is a critical point of } f|V \} \) is semialgebraic by the Tarski-Seidenberg theorem \([M]\), since \( A = V \setminus \pi(TV \setminus \ker Df) \), where \( \pi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

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is the first projection, and \( TV \) and \( \ker Df \) are semialgebraic in \( \mathbb{R}^n \times \mathbb{R}^n \). Therefore, \( A \) has finitely many connected components [G]. Since \( f|A \) is locally constant, it is constant in each connected component. □

Assume that \( f : (\mathbb{R}^n; 0) \to (\mathbb{R}; 0) \) is a polynomial. Hence \( V(f) \) admits a canonical Whitney stratification in finitely many connected strata, each of which is a semialgebraic submanifold of \( \mathbb{R}^n \). By the above theorem there exists an \( \epsilon_0 > 0 \) such that \( \forall \epsilon, 0 < \epsilon < \epsilon_0 \Rightarrow S_\epsilon \cap V_i, \forall i \). This proves

1.2. PROPOSITION. For \( 0 < \epsilon < \epsilon_0 \), \( V(f) \cap B_\epsilon \) has finitely many connected components \( K_i \), each one semialgebraic and adherent to the origin.

We proceed to define the germs \( \phi_i : (\mathbb{R}^+; 0) \to (\mathbb{R}; 0) \) whose behavior will detect the existence of an extremum of \( f \).

Corresponding to each \( K_i \), let \( \phi_i : (\mathbb{R}^+; 0) \to (\mathbb{R}; 0) \) be defined by \( \phi_i(0) = 0 \) and, for \( 0 < \epsilon < \epsilon_0 \), \( \phi_i(\epsilon) = f(x) \), where \( x \in K_i \cap S_\epsilon \). \( \phi_i \) is well defined since \( f|K_i \cap S_\epsilon \) is constant, because \( K_i \cap S_\epsilon \) is connected and \( \text{grad}(f|K_i \cap S_\epsilon) = 0 \).

We have thus associated to the polynomial \( f \) a finite number of germs \( \phi_i \) of a single variable.

1.3. THEOREM. (a) A necessary and sufficient condition for \( f \) to admit a minimum (a strong minimum, a maximum, a strong maximum) at \( 0 \in \mathbb{R}^n \) is that the same holds true for each \( \phi_i \) at \( 0 \in \mathbb{R}^+ \).

(b) A necessary and sufficient condition for \( f \) not to admit an extremum at \( 0 \in \mathbb{R}^n \) is that there exist \( i, j \) such that \( \phi_i \) has a strong maximum and \( \phi_j \) a strong minimum at \( 0 \in \mathbb{R}^+ \).

PROOF. We prove only that the condition of (a) is sufficient; the rest follows easily. Suppose, for example, that all \( \phi_i \) have a minimum at \( 0 \in \mathbb{R}^+ \). Since we have a finite number of \( \phi_i \), there exists an \( \epsilon_1 > 0 \) (\( \epsilon_1 \leq \epsilon_0 \)) such that \( \forall \epsilon > 0, \epsilon < \epsilon_1, \forall i \phi_i(\epsilon) \geq 0 \). The set of the critical points of \( f|S_\epsilon \) is \( V(f) \cap S_\epsilon \). Therefore the minimum value of \( f|S_\epsilon \) is \( \phi_i(\epsilon) \) for some \( i \), which shows that \( f|B_{\epsilon_1} \geq 0 \), so \( f \) has a minimum at \( 0 \in \mathbb{R}^n \). The other cases may be treated similarly. □

1.4. REMARK. \( K_i \) is semialgebraic and \( 0 \in K_i \). Hence, by the curve selection lemma [M], there exists an analytic curve \( \gamma_i : [0, 1) \to \mathbb{R}^n \) such that \( \gamma_i(0) = 0 \) and \( \gamma_i(t) \in K_i \) for \( t \neq 0 \). Note that

\[
\gamma_i(t) = a_{m_i} t^{m_i} + \sum_{k=m_i+1}^{\infty} a_k t^k
\]

with \( a_{m_i} \neq 0, m_i \geq 1 \) and \( a_k \in \mathbb{R}^n \). Thus \( h_i = |\gamma_i| \) is a germ of homeomorphism at \( 0 \in \mathbb{R}^+ \) because \( h_i \) is continuous and locally strictly increasing. Since \( f \circ \gamma_i = \phi_i \circ h_i \) and \( f \circ \gamma_i \) is analytic, we have \( \phi_i \equiv 0 \), or \( 0 \) is an isolated zero of \( \phi_i \), and so a strong extremum. Moreover, since \( h_i \) is increasing, if \( \phi_i \) has a minimum (maximum), so has \( f \circ \gamma_i \), and conversely. This fact can be detected by the sign of the least nonzero derivative of \( f \circ \gamma_i \) at \( 0 \). Furthermore, we have

\[
h_i(t) = |a_{m_i}| t^{m_i} (1 + \psi(t))^{1/2}
\]

for some analytic function \( \psi(t) \) with \( \psi(0) = 0 \). Then

\[
(|a_{m_i}|/2)t^{m_i} \leq h_i(t) \leq 2|a_{m_i}|t^{m_i},
\]
or, equivalently,

\[(a_m)|/2| h_i^{-1}(u)|^{m_i} \leq u \leq 2a_m|| h_i^{-1}(u)|^{m_i} \].

These inequalities will be used in §3.

2. Generalized jets.

2.0. Definition. We say that the germ \( f : (\mathbb{R}^n; 0) \to (\mathbb{R}; 0) \) has punctual \( k \)-jet at \( 0 \in \mathbb{R}^n \) if there is polynomial \( P : \mathbb{R}^n \to \mathbb{R} \) of degree less than \( k \) \((\leq k, \text{deg } P \leq -\infty)\) such that \( \lim_{x \to 0} ((f(x) - P(x))/|x|^k) = 0 \).

Obviously such a polynomial, if it exists, is unique and is the Taylor polynomial of \( f \) when \( f \) has one. We denote it by \( j^k f \) and call it the \( k \)-jet of \( f \).

The set of all germs \( f : (\mathbb{R}^n; 0) \to (\mathbb{R}; 0) \) which have a \( k \)-jet will be denoted by \( \mathcal{G}^k(n) \).

Note that \( \mathcal{C}^k(n) \subset \mathcal{G}^k(n) \subset \mathcal{G}^{k-1}(n) \). These inclusions are trivial and strict since the function

\[ f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ x^2 \exp(-x^{-2}), & x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases} \]

is discontinuous except at 0 and belongs to \( \mathcal{G}^k(1) \) for any \( k \).

2.1. Remarks. The concept of \( k \)-jet has a natural extension to germs \( f : (\mathbb{R}^n; 0) \to (\mathbb{R}^p; 0) \) and, in this context, the general theorems of the usual differential calculus, such as the chain-rule, the local forms of immersions and submersions, Morse’s lemma, etc.

As illustrations we present the local form of submersions, when \( p = 1 \), and the splitting lemma.

2.2. Local form of submersions. Let \( f \in \mathcal{G}^k(n) \), \( k \geq 1 \). If \( j^1 f \neq 0 \), then for all \( r \) and \( s \) \((1 \leq r \leq k; r \leq s \leq \omega)\) there exists a germ of diffeomorphisms \( \phi : (\mathbb{R}^n; 0) \to (\mathbb{R}^n; 0) \) of class \( C^\infty \) such that \( j^r(f \circ \phi) = \pi_1 \).

Proof. It is enough to consider the case \( r = k \), \( s = \omega \). Since \( j^k f \) is a polynomial, \( j^1(j^k f) = j^1 f \neq 0 \), and therefore there exists \( \phi \in \text{Diff}^\omega(\mathbb{R}^n; 0) \) such that \( (j^k f)\circ \phi = \pi_1 \). Hence

\[ j^k(f \circ \phi) = j^k((j^k f) \circ (j^k \phi)) = j^k((j^k f) \circ \phi) = j^k \pi_1 = \pi_1. \]

2.3. Splitting lemma. Let \( f \in \mathcal{G}^k(n) \), \( k \geq 2 \). If \( j^1 f = 0 \) then there exist a splitting \( \mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^v \times \mathbb{R}^w \) and, for all \( r \) and \( s \) \((2 \leq r \leq k; r \leq s \leq \infty)\), a germ of diffeomorphism \( \phi : (\mathbb{R}^n; 0) \to (\mathbb{R}^n; 0) \) of class \( C^\infty \) such that \( j^r(f \circ \phi)(x; y; z) = |x|^2 - |y|^2 + p_r(z) \), where \( p_r(z) \) is a polynomial of degree \( \leq r \) satisfying \( j^2 p_r = 0 \).

Proof. We consider only the case \( r = k \), \( s = \infty \). Using the fact that \( j^1(j^k f) = 0 \) and the classical splitting lemma for the polynomial \( j^k f \), we can find a splitting \( \mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^v \times \mathbb{R}^w \) and a \( \phi \in \text{Diff}^\infty(\mathbb{R}^n; 0) \) such that

\[ ((j^k f) \circ \phi)(x; y; z) = |x|^2 - |y|^2 + g(z), \]

where \( j^2 g = 0 \). But \( j^k(f \circ \phi) = j^k((j^k f) \circ \phi) \), which implies

\[ j^k(f \circ \phi)(x; y; z) = |x|^2 - |y|^2 + j^k g(z). \]

3. Jet-detectable extrema. Now we extend Theorem 1.3, with the obvious adaptations, to functions which have generalized \( k \)-jets.
3.0. Definition. For \( f \in \mathcal{G}(n) \) we say that \( f \) is \( k \)-decidable (\( k \leq r \)) if it is possible, using only \( j^k f \), to decide if the origin is a maximum, a minimum, or not an extremum of \( f \).

Note that if \( f \) is \( k \)-decidable and \( k \leq s \leq r \), then \( f \) is \( s \)-decidable.

Let \( f \in \mathcal{G}(n) \). We fix a \( k \leq r \) and apply to the polynomial \( j^k f \) the process described before. We get a finite family of germs \( \phi_i : (\mathbb{R}_+; 0) \to (\mathbb{R}; 0) \).

We now proceed to define the germ \( f_i : (\mathbb{R}_+; 0) \to (\mathbb{R}; 0) \) whose behavior at \( 0 \in \mathbb{R}_+ \) will possibly allow us to decide about the existence of an extremum of \( f \) at \( 0 \in \mathbb{R}^n \).

With the notation of 1.4 we have \( \phi_i \circ h_i = (j^k f) \circ \gamma_i \), which is analytic, and

\[
\gamma_i = a_m t^{m_i} + \sum_{j=m_i+1}^{\infty} a_j t^j,
\]

where \( a_m \neq 0 \). We put \( f_i = j^{m_i k}(\phi_i \circ h_i) \).

The germ \( f_i \) so defined is the natural substitute for the \( k \)-jet of \( \phi_i \) when \( \phi_i \) does not have such a jet. We remark that the theorem below holds if we replace \( f_i \) by \( j^k \phi_i \) when this latter exists.

3.1. Theorem. (a) If all \( f_i \) have a strong minimum (maximum) at \( 0 \in \mathbb{R}_+ \), so does \( f \) at \( 0 \in \mathbb{R}^n \).

(b) If among the \( f_i \) there is one which has a strong minimum and another which has a strong maximum at \( 0 \in \mathbb{R}_+ \), then \( f \) does not have an extremum at \( 0 \in \mathbb{R}^n \).

(c) In the remaining cases, \( f \) is not \( k \)-decidable.

Proof. (a) We consider only the minimum case. Since \( f_i \) is a polynomial of degree \( \leq m_i k \) we can choose \( \alpha_i > 0 \) and \( \tau_i > 0 \) such that \( \forall t \),

\[
0 < t < \tau_i \Rightarrow f_i(t) > \alpha_i t^{m_i k} > \beta_i |h_i(t)|^k,
\]

where \( \beta_i > 0 \) is obtained by using inequalities 1.4. Since the family is finite we may take \( \beta > 0 \) and \( \tau > 0 \) such that \( \forall i, \forall t, 0 < t < \tau \Rightarrow f_i(t) > \beta |h(t)|^k \).

We also have \( \forall \lambda > 0, \exists \tau > 0 \) such that \( \forall i \tau \geq h_i(\tau) \) and \( \forall u \)

\[
0 < u < \tau \Rightarrow |\phi_i(u) - f_i[h_i^{-1}(u)]| < \lambda |h_i^{-1}(u)|^{m_i k} < \beta u^k/2
\]

for \( \lambda \) small enough.

Finally, we choose \( \varepsilon > 0, \varepsilon < \varepsilon \forall x, |x| = \delta < \varepsilon \Rightarrow |f(x)j^k f(x)| < \beta \delta^k/4 \). So, we have \( \forall x \)

\[
|x| = \delta < \varepsilon \Rightarrow f(x) \geq \min j^k f|S_\delta - \beta \delta^k/4.
\]

By Theorem 1.3 \( \exists \delta |j^k f|S_\delta = \phi_i(\delta) \), therefore \( \forall x \)

\[
|x| = \delta < \varepsilon \Rightarrow f(x) \geq \phi_i(\delta) - \beta |\delta|^k/4 \geq \beta \delta^k/4,
\]

which shows (a).

(b) We take \( \alpha, \tau > 0 \) such that \( \forall t \)

\[
0 < t < \tau \Rightarrow f_i(t) > \alpha t^{m_i k} > \beta_i |h_i(t)|^k,
\]

and

\[
f_j(t) < -\alpha t^{m_j k} < -\beta_j |h_j(t)|^k.
\]
Let $\beta = \min(\beta_i, \beta_j) > 0$ and $\zeta > 0$, $\zeta < h_i(\tau)$ and $\zeta < h_j(\tau)$ such that $\forall u,$

$$0 < u < \zeta \Rightarrow |\phi_r(u) - f_r(h_i^{-1}(u))| < \beta u^k/2, \quad r = i, j.$$ 

Finally we take $\varepsilon > 0$, $\varepsilon < \zeta$ such that $\forall x,$

$$|x| < \varepsilon \Rightarrow |f(x) - j^k f(x)| < \beta \varepsilon^k/4.$$ 

We claim that for any $\delta$, $0 < \delta < \varepsilon$, $f|S_\delta \cap K_i > 0$ and $f|S_\delta \cap K_j < 0$. For, if $x_j \in S_\delta \cap K_j$,

$$f(x_j) \leq \phi(|x_j|) + \frac{\beta}{4} |x_j|^k \leq f_j[h_j^{-1}(|x_j|)] + \left(\frac{\beta}{2} + \frac{\beta}{4}\right) |x|^k \leq -\frac{\beta}{4} |x_j|^k < 0.$$ 

Similarly, if $x_i \in S_\delta \cap K_i$,

$$f(x_i) \geq \beta |x_i|^k/4 > 0,$$

which proves (b).

(c) Suppose, for example, that all $f_j$ have a minimum but $f_j$ has a weak minimum at $0 \in \mathbb{R}^+$ (hence $f_j \equiv 0$). It is enough to construct two germs, $g_0$ and $g_1$, having the same $k$-jet as $f$ such that $0 \in \mathbb{R}^n$ is a strong minimum for $g_0$ and is not a minimum (not even a weak one) for $g_1$. To do that, for each $j$, such that $f_j \equiv 0$, we choose $\varepsilon_j > 0$ (small enough) and $\alpha_j > 0$ (big enough) in order that $\forall t$,

$$0 < t < \varepsilon_j \Rightarrow |\phi_j(h_j(t))| < \alpha_j t^{m_j k+1} < \beta_j[h_j(t)]^{k+(1/m_j)}$$

($\beta_j$ as in 1.4). Since there are finitely many such $j$'s we take $\varepsilon, \beta > 0$, $m \in \mathbb{N}^*$ such that for all those $j$'s we have $\forall \varepsilon$,

$$0 < \varepsilon \Rightarrow |\phi_j(h_j(t))| < \beta[h_j(t)]^{k+(1/m)}$$

and put

$$g_\nu(x) = j^k f(x) - (-1)^\nu \beta |x|^{k+1/m}, \quad \nu = 0, 1.$$ 

It is clear that $j^k g_\nu = j^k f$ and the critical points of $g_\nu|S_\delta$ are precisely those of $j^k f|S_\delta$, $\forall \delta > 0$. Moreover, $\forall i$,

$$g_\nu(\gamma_i(t)) = \phi_i(h_i(t)) - (-1)^\nu \beta[h_i(t)]^{k+1/m}.$$ 

Let us show that $0 \in \mathbb{R}^n$ is not even a weak minimum for $g_0$. For if $f_j \equiv 0$ and $x_i \in K_j$ with $|x_j| < h_j(\varepsilon)$, then

$$g_0(x_j) = g_0(\gamma_j(t)) = \phi_j(h_j(t)) - \beta[h_j(t)]^{k+1/m} < 0.$$ 

Let us show finally that $0 \in \mathbb{R}^n$ is a strong minimum for $g_1$. Let $\delta > 0$ such that $\forall j$, $f_j \equiv 0 \Rightarrow \delta < h_j(\varepsilon)$. We have min $g_1|S_\delta = g_1(\gamma_i(t))$ for some $i$, and then

$$\min g_1|S_\delta = \phi_i(h_i(t)) + \beta[h_i(t)]^{k+1/m}.$$ 

If $f_i \equiv 0$ then $\beta[h_i(t)]^{k+1/m} > 0$, otherwise $\phi_i(h_i(t)) > \mu_i t^k$ for $\mu_i$ small enough and any $t$ nearby 0, and again $g_1(\gamma_i(t)) > 0$. □

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