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NORMAL SUBGROUPS OF $\mathrm{Diff}^{\Omega}(\mathbf{R}^3)$

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ABSTRACT. Let Ω be a volume element on $\mathbf{R}^3$ of infinite total Ω-volume. We denote by $\mathrm{Diff}^{\Omega}(\mathbf{R}^3)$ the group of all Ω-preserving diffeomorphisms of $\mathbf{R}^3$, by $\mathrm{Diff}_{\mathrm{c}}^{\Omega}(\mathbf{R}^3)$ the subgroup of all elements with compact support and by $\mathrm{Diff}_{\mathrm{f}}^{\Omega}(\mathbf{R}^3)$ the subgroup of all elements whose support has finite Ω-volume.

We prove that there is no normal subgroup between $\mathrm{Diff}_{\mathrm{c}}^{\Omega}(\mathbf{R}^3)$ and $\mathrm{Diff}_{\mathrm{f}}^{\Omega}(\mathbf{R}^3)$.

In my paper *Normal subgroups of* $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ [**5**], I studied the normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbf{R}^n)$ for $n \geq 4$, Ω being any volume element on $\mathbf{R}^n$. All results in the paper hold for $n = 3$ except Lemma 4.4. Thus we know that the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^3)$ of all elements compactly Ω-isotopic to the identity, $\mathrm{Diff}_{\mathrm{co}}^{\Omega}(\mathbf{R}^3)$, is simple, and there is a maximal proper normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbf{R}^3)$, $\mathrm{Diff}_{W}^{\Omega}(\mathbf{R}^3)$, the subgroup of all elements with set of nonfixed points of finite Ω-volume.

The purpose of this paper is to prove a modification of Lemma 4.4 of [**5**] for $n = 3$, getting, as a consequence, that there is no normal subgroup between $\mathrm{Diff}_{\mathrm{c}}^{\Omega}(\mathbf{R}^3)$ and $\mathrm{Diff}_{\mathrm{f}}^{\Omega}(\mathbf{R}^3)$.

The importance of Lemma 4.4 is given by the fact that the basic method for understanding the normal subgroups is to factor a diffeomorphism into a product of diffeomorphisms whose support is well-controlled and then to manipulate this support using techniques of [**2**].

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Let us start by giving some definitions on infinite links.

DEFINITION. Let $\coprod_{i\geq 1} \alpha_i$, $\coprod_{i\geq 1} \beta_i$ be two locally finite sets of disjoint smooth paths in $\mathbf{R}^3$ such that $\alpha_i \cap \beta_j = \varnothing$ if $i \neq j$ and

$$\alpha_i \cap \beta_i = (\alpha_i(0) = \beta_i(0)) \cup (\alpha_i(1) = \beta_i(1)).$$

Let p: $\mathbf{R}^3 \to \mathbf{R}^2 \times \{0\}$ given by $p(x,y,z) = (x,y,0)$ be the parallel projection.

We call a crossing of the link $L = \coprod_{i\geq 1} \alpha_i \cup \coprod_{i\geq 1} \beta_i$ the set of points $p^{-1}(c)$, where c is a multiple point of $p|_L$. When no confusion is possible we also call the point c a crossing.

Since every differentiable knot is equivalent to one in regular position, and since in L we have a locally finite sequence of differentiable paths, we can assume that all crossings are double. Let c be a double point of $p|_L$. We call c' the point of $p^{-1}(c)$ with larger z-coordinate and c'' the other one.

Now, we have two different types of crossings:

(a) $p^{-1}(c) \subset \alpha_i \cup \alpha_j$ or $p^{-1}(c) \subset \beta_i \cup \beta_j$;

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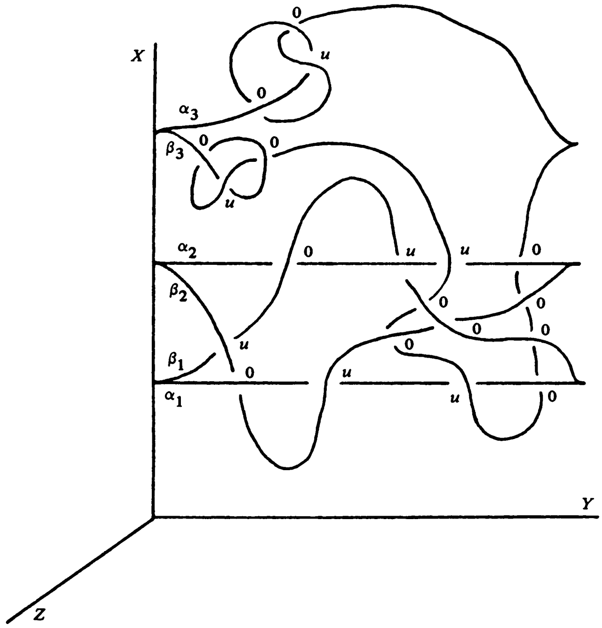

FIGURE 1

(b) one point of $p^{-1}(c)$ lies in α_i and the other in β_j.

DEFINITION. A crossing $p^{-1}(c)$ is an overcrossing and we denote it by "O" in the following cases:

(i) Type (a): if c' lies in α_i when $i < j$ or if we first find c' when α_i is traversed from $\alpha_i(0)$ to $\alpha_i(1)$ if $i = j$; similarly if $p^{-1}(c) \subset \beta_i \cup \beta_j$.

(ii) Type (b): if c' lies in α_i when $i \leq j$ or in β_j when $j < i$.

Otherwise, we call a crossing an undercrossing and we denote it by "U".

We now prove

LEMMA. *Let L be as above. There are smooth paths $\coprod_{i\geq 1} \alpha'_i$, $\coprod_{i\geq 1} \beta'_i$ such that α'_i is very near α_i and β'_i is very near β_i, $\alpha'_i \cap \beta'_j = \emptyset$ if $i \neq j$, $\alpha'_i \cap \beta'_i = (\alpha'_i(0) = \beta'_i(0)) \cup (\alpha'_i(1) = \beta'_i(1))$, and all crossings of $(\coprod_{i\geq 1} \alpha'_i) \cup (\coprod_{i\geq 1} \beta'_i)$ are overcrossings.*

PROOF. We define α'_i, β'_i inductively on i.

α'_i, β'_i are different from α_i, β_i only in a chosen neighbourhood of each undercrossing $U = p^{-1}(c)$ where α'_i and β'_i are defined as follows.

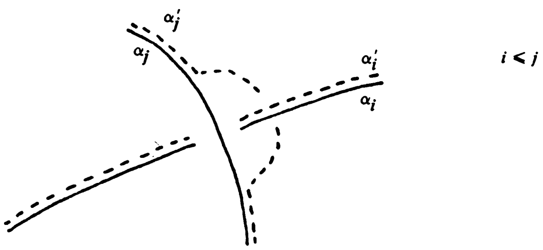

FIGURE 2

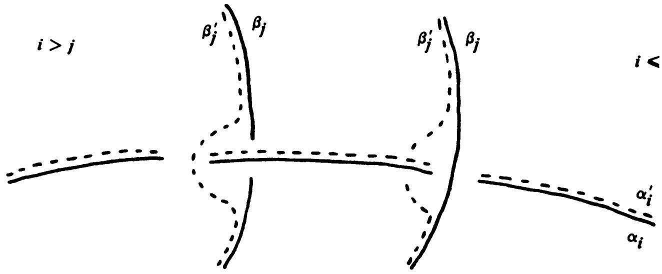

FIGURE 3

(I) U is of type (a). On a neighbourhood of c', α_j' (resp. β_j') goes vertically (in the z-direction) under α_i (resp. β_i) instead of over. On a neighbourhood of c'', α_i' (resp. β_i') is the same as α_i (resp. β_i) (see Figure 2).

(II) U is of type (b). α_i' is α_i. On a neighbourhood of c', β_j' goes vertically (in the z-direction) under α_i instead of over it if $i \leq j$; if $i > j$, on a neighbourhood of c'', β_j' goes vertically (also in the z-direction) over α_i instead of under.

Thus, all crossings of $\coprod_{i\geq 1} \alpha_i' \cup \coprod_{i\geq 1} \beta_i'$ are overcrossings.

REMARK. We know by McDuff [**6**] that the loops $\alpha_i \cup \alpha_i'$ and $\beta_i \cup \beta_i'$ are both unknotted for any i.

Furthermore, notice that the infinite link $\coprod_{i\geq 1} \alpha_i' \cup \coprod_{i\geq 1} \beta_i'$ constructed above is untangled in the sense that it is diffeomorphic to the standard one.

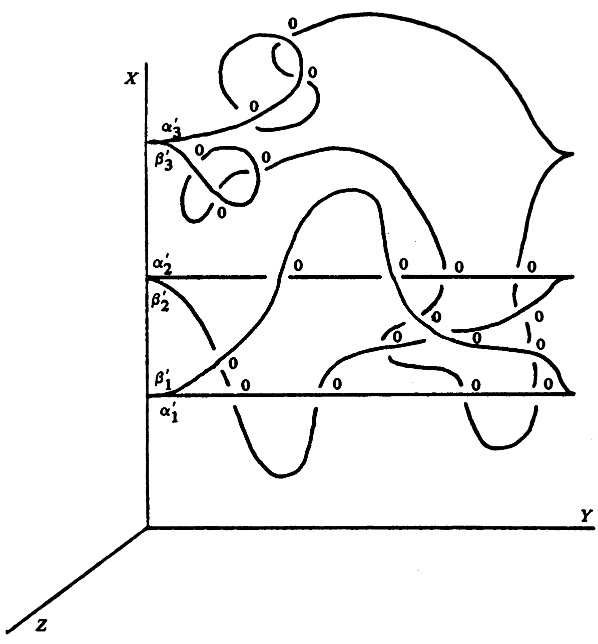

FIGURE 4

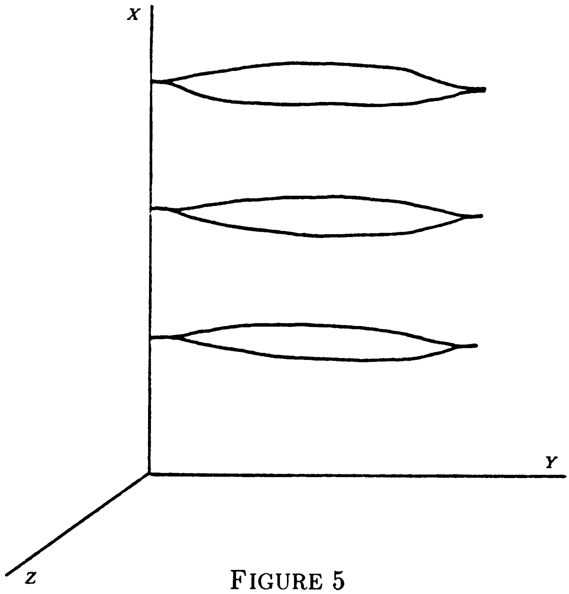

FIGURE 5

Before proving Lemma 4.4 for $n = 3$ we will define a strip.

DEFINITION. A strip in $\mathbf{R}^3$ is the image under some diffeomorphism of $\mathbf{R}^3$, g, of the tube $\{(x, y, z) \in \mathbf{R}^3\colon\ x \geq 0,\ y^2 + z^2 \leq 1\}$.

Notice that a strip may have finite Ω-volume since g may not be volume preserving.

We now state and prove Lemma 4.4 for $n = 3$.

THEOREM. *Let f be any volume element of* $\mathrm{Diff}_{\mathrm{f}}^{\Omega}(\mathbf{R}^3)$ *with support in a strip V of infinite Ω-volume. Then $f = f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6$, where f_i lies in* $\mathrm{Diff}_{\mathrm{f}}^{\Omega}(\mathbf{R}^3)$ *and has support in a strip V_i of finite Ω-volume.*

PROOF. As in Lemma 4.4 of [**5**] we get a disjoint union of closed balls $\coprod_{i\geq 1} B_i \subset \operatorname{int} V - \operatorname{supp} f$ such that $\mathrm{vol}_\Omega(V - \coprod_{i\geq 1} B_i) < \infty$. We can join each ball B_i to ∂V by an unknotted smooth path α_i in V satisfying:

(i) The set $\{\alpha_i\}$ is locally finite.

(ii) $\alpha_i \cap \alpha_j = \emptyset$ if $i \neq j$.

(iii) $\alpha_i \cap \beta_j = \emptyset$ if $i \neq j$ and $\alpha_i \cap B_i = \alpha_i(1)$. Also, we can get f_1, a volume preserving diffeomorphism with support in a strip of finite Ω-volume such that $f_1^{-1} \circ f(\alpha_i) \cap \alpha_j = \emptyset$ for any $i \neq j$ and $f_1^{-1} \circ f(\alpha_i)$ and α_i only meet on a connected neighbourhood of its endpoints.

We consider now the infinite link $L = \coprod_{i\geq 1} \alpha_i \cup \coprod_{i\geq 1} \beta_i$, where $\beta_i = f_1^{-1} \circ f(\alpha_i)$, and we apply the Lemma to it. So we get, for any i, $\alpha_i' = \alpha_i$ because the α_i never cross each other and $\coprod_{i\geq 1} \beta_i'$, where β_i' is different from $f_1^{-1} \circ f(\alpha_i)$ only in a small neighbourhood of each undercrossing. $\coprod_{i\geq 1} \alpha_i \cup \coprod_{i\geq 1} \beta_i'$ is untangled and for any i, $\alpha_i \cup \beta_i'$ and $\beta_i \cup \beta_i'$ are both unknotted.

Let β_i'' be the path $f_1^{-1} \circ f(\alpha_i)$ except near an undercrossing of type (b) where we have changed it to an overcrossing as in the Lemma. So there is a volume preserving diffeomorphism, f_2, with support in a disjoint union of cells of Ω-volume as small as we like such that $f_2^{-1}(\beta_i) = f_2^{-1} \circ f_1^{-1} \circ f(\alpha_i) = \beta_1''$.

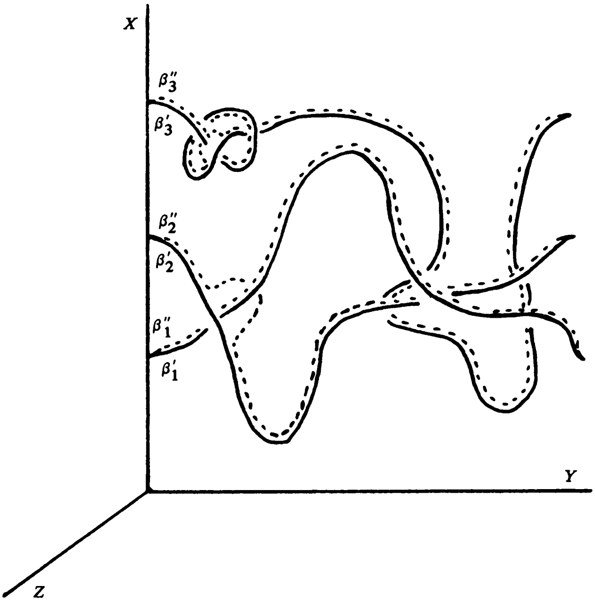

FIGURE 6

Now we consider the link $\coprod_{i\geq 1}\beta_i' \cup \coprod_{i\geq 1}\beta_i''$. In the same way as in [**6**, Lemma 8], we can prove that the link $\coprod_{i\geq 1}\beta_i' \cup \coprod_{i\geq 1}\beta_i''$ is untangled, therefore, there is a volume preserving diffeomorphism, f_3^{-1}, with support in a disjoint union of cells of Ω-volume as small as we like such that $f_3^{-1}(\beta_i'') = \beta_i'$ for any i.

Now we can construct, inductively, pairwise disjoint embedded 2-dimensional open discs E_i such that $\partial\overline{E}_i = \alpha_i \cup \beta_i'$ for any i. Also, there are smooth unknotted paths γ_i in $V - \coprod_{i\geq 1} B_i - \coprod_{i\geq 1}\overline{E}_i$ joining $\alpha_i(0)$ and $\alpha_i(1)$, near α_i and such that each crossing of $\coprod_{i\geq 1}\gamma_i \cup \coprod_{i\geq 1}\beta_i'$ is an overcrossing. Thus, there are pairwise disjoint small neighbourhoods U_i of $\overline{E}_i$ in $V - \coprod_{i\geq 1} B_i - \coprod_{i\geq 1}\gamma_i$. Then, there is an isotopy θ: $\coprod_{i\geq 1}\alpha_i \times [0,1] \to \coprod_{i\geq 1} U_i$ with θ_o equal to the identity and θ_1 equal to $f_3^{-1}\circ f_2^{-1}\circ f_1^{-1}\circ f$.

Now, the proof follows as in Lemma 4.4 of [**5**].

COROLLARY. *There is no normal subgroup between* $\mathrm{Diff}_c^{\Omega}(\mathbf{R}^3)$ *and* $\mathrm{Diff}_f^{\Omega}(\mathbf{R}^3)$.

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