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SHORTER NOTES

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TWO EASY EXAMPLES OF ZERODIMENSIONAL SPACES

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ABSTRACT. We present simple examples of a realcompact zerodimensional space which is not N-compact and an N-compact space which is not strongly zerodimensional.

Let N be the usual space of natural numbers. A topological space is called N-*compact* if it is homeomorphic to a closed subspace of some power of N. It is well known that realcompact strongly zerodimensional$\Rightarrow$ N-compact $\Rightarrow$realcompact and zerodimensional. The converses are false, but known counterexamples are fairly complicated (cf. [**1**, **5**–**10**]). The aim of this note is to present both counterexamples in a unified and simple way. It should be mentioned, however, that the spaces we construct are neither normal nor metacompact, and therefore they do not have any of the additional properties which the previously known examples have.

Our construction has evolved from the ideas given in [**2** and **7**]. As the referee has pointed out, the technique we use is close to that used by M. Wage in [**11**] in order to construct two strongly zerodimensional spaces whose product is not strongly zerodimensional. In fact, it turns out that our technique, although used for different purposes, is basically the same as that developed in [**11**].

All undefined notions can be found in [**3** and **4**]. We shall use in the sequel the ultrafilter characterization of N-compactness which states that a zerodimensional space X is N-compact if and only if every clopen ultrafilter on X with the countable intersection property has nonempty intersection.

The construction. Consider an arbitrary separable metric space X of cardinality $\mathfrak{c}$. Let D be an arbitrary countable dense subset of X. For every subset A of D with $|\mathrm{cl}_X A \cap \mathrm{cl}_X(D-A)| = \mathfrak{c}$ choose a point x_A in $\mathrm{cl}_X A \cap \mathrm{cl}_X(D-A)$ in such a way that $x_A \neq x_{A'}$ whenever $A \neq A'$. Moreover, for every x from $X - D$ let $S(x)$ be a sequence in D converging to x such that if $x = x_A$ for some $A \subset D$ then both A and $D - A$ contain infinitely many points from $S(x)$. Define a new topology on X by agreeing that all points from D are isolated and neighborhoods of $x \in X - D$ contain all but finitely many points from $S(x)$. Denote the obtained space by X^*.

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The topology of X^* is finer than that of X and has a base consisting of clopen countable compact sets. It follows that X^* is a realcompact (cf. [**4**, Chapters 8.18 and 15.24]) zerodimensional locally countable and locally compact space. Clearly the set D is dense in X^*. The following is the crucial property of X^*:

> $(*)$ if U is a clopen subset of X^* then its boundary in X has cardinality less than $\mathfrak{c}$.

Indeed, it follows immediately from the definition of X^* that $|\mathrm{cl}_X(U \cap D) \cap \mathrm{cl}_X(D - U)| < \mathfrak{c}$. On the other hand, since D is dense in X^* we have $U \subset \mathrm{cl}_{X^*}(U \cap D) \subset \mathrm{cl}_X(U \cap D)$ and $X - U \subset \mathrm{cl}_{X^*}(D - U) \subset \mathrm{cl}_X(D - U)$. Hence $\mathrm{cl}_X U = \mathrm{cl}_X(U \cap D)$, $\mathrm{cl}_X(X - U) = \mathrm{cl}_X(D - U)$ and $|\mathrm{cl}_X U \cap \mathrm{cl}_X(X - U)| < \mathfrak{c}$.

EXAMPLE 1. Let X be the Euclidean plane. We prove that the realcompact zerodimensional space X^* is not N-compact. To this end, observe that if U is a subset of the Euclidean plane with a small boundary, i.e. $|\mathrm{cl}_X U \cap \mathrm{cl}_X(X - U)| < \mathfrak{c}$, then either $\mathrm{cl}_X U = X$ and $|X - U| < \mathfrak{c}$ or $\mathrm{cl}_X(X - U) = X$ and $|U| < \mathfrak{c}$. This and property $(*)$ imply that the family

$$\xi = \{U \subset X\colon\ U \text{ is clopen in } X^* \text{ and } |X - U| < \mathfrak{c}\}$$

forms a clopen ultrafilter on X^* with the countable intersection property. Since every point from X^* has a countable clopen neighborhood, the ultrafilter ξ has an empty intersection.

EXAMPLE 2. Let X be the subspace of the separable Hilbert space $(l_2, \|\cdot\|)$ consisting of all points with rational coordinates. We prove that X^* is N-compact but not strongly zerodimensional.

To see the former, consider an arbitrary clopen ultrafilter ξ on X^* with the countable intersection property. The family $\eta = \{U \in \xi\colon\ U \text{ is clopen in } X\}$ forms a clopen ultrafilter on X with the countable intersection property. Since X is a Lindelöf space there is a point x in $\bigcap \eta$. Notice that every point in X is the intersection of countably many clopen subsets of X. Hence $\{x\}$ is the intersection of countably many members of ξ and, therefore, $x \in \bigcap \xi$.

To see that X^* is not strongly zerodimensional, it suffices to observe (compare [**3**, Example 6.2.19]) that every subset of X containing the closed set $Z_1 = \{x \in X\colon\ \|x\| \leq 1\}$ and disjoint from the closed set $Z_2 = \{x \in X\colon\ \|x\| \geq 2\}$ has a boundary of cardinality $\mathfrak{c}$. This and property $(*)$ imply that the disjoint zerosets Z_1 and Z_2 in X^* cannot be separated by any clopen subset of X^*.

REMARK. If we change the topology of X^* from Example 1 by assuming that for a fixed point from $X - D$ the neighborhood system consists of sets open in the Euclidean plane, then the resulting space is completely regular and scattered but not zerodimensional.

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