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AN ELEMENTARY TRANSFORMATION OF A SPECIAL UNIMODULAR VECTOR TO ITS TOP COEFFICIENT VECTOR

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ABSTRACT. Let R be a commutative ring, $\mathbf{v}(X)$ a unimodular n-vector ($n \geqslant 3$) over $R[X]$. Suppose the leading coefficients in $\mathbf{v}(X)$ form a unimodular vector $L(\mathbf{v})$ over R. Then some element in $E_n(R[X])$ will transform $\mathbf{v}(X)$ to $L(\mathbf{v})$.

1. Introduction and motivation. For a commutative ring R, $E_n(R)$ will denote the subgroup of the determinant one matrices $\mathrm{Sl}_n(R)$ generated by $E_{ij}(\lambda) = I_n + \lambda e_{ij}$, where I_n denotes the $n \times n$ identity matrix, $1 \leqslant i \neq j \leqslant n$, $\lambda \in R$, and e_{ij} is the matrix whose (i, j)th entry is 1 and all other entries are zeros. If $\alpha \in E_n(R)$, call it an elementary matrix.

A vector $\mathbf{v} = (v_1,\ldots,v_n) \in R^n$ is called *unimodular* if there is a vector $\mathbf{w} = (w_1,\ldots,w_n) \in R^n$ with $\mathbf{v} \cdot \mathbf{w}^t = \sum_i v_i w_i = 1$. $\mathrm{Um}_n(R)$ will denote the set of all unimodular vectors in R^n. Clearly, $E_n(R)$ acts on $\mathrm{Um}_n(R)$. Denote by '$\equiv$ (mod $E_n(R)$)' vectors in the same orbit under this action.

For a polynomial $f \in R[X]$, $L(f)$ will denote the leading coefficient of f.

A vector $\mathbf{v} = (v_1,\ldots,v_n) \in \mathrm{Um}_n(R[X])$ is called a *special unimodular* vector if $L(\mathbf{v}) = (L(v_1),\ldots,L(v_n)) \in \mathrm{Um}_n(R)$.

The principal aim of this short note is to point out

THEOREM. *Let* $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$, *be a special unimodular vector. Then* $\mathbf{v} \equiv L(\mathbf{v}) \pmod{E_n(R[X])}$.

This simplifies and extends a result of A. A. Suslin in [S, Proposition 2.4].

The theorem settles affirmatively a question raised by M. P. Murthy during discussions, viz. if $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$, has a *monic* polynomial as one of its entries, then is $\mathbf{v} \equiv \mathbf{e}_1 = (1,0,\ldots,0) \pmod{E_n(R[X])}$?

Murthy's question was raised in relation to the following problem on the efficient generation of ideals in polynomial rings.

PROBLEM. *Let R be a noetherian ring, and I an ideal in the polynomial ring $R[X]$ containing a monic polynomial. Let I satisfy the following condition on its conormal bundle; viz the R/I-module I/I^2:*

$$\mu_{R/I}(I/I^2) \overset{\text{def}^n}{=} \text{the least number of generators of } I/I^2 \text{ as a } R/I\text{-module} \geqslant \dim(R[X]/I) + 2.$$

Then is $\mu(I) = \mu(I/I^2)$?

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Recently, S. Mandal [**M**] settled the above problem affirmatively. Earlier, N. Mohan Kumar's reductions in [**MK**] had established a surjection $P \to I \to 0$ with P a projective $R[X]$-module of rank $\mu(I/I^2)$. By utilizing the finer information available now by our theorem, we can considerably simplify the treatment in [**MK**] and show that I is the onto image of a *free* module of rank $\mu(I/I^2)$, thereby providing a quick alternative solution to the above problem.

Let me mention two applications of this problem.

COROLLARY 1. *Let R be a noetherian ring, and I an ideal in $R[X_1,\ldots,X_n]$, with* height $I > \dim R$, *and* $\mu(I/I^2) \geqslant \dim(R[X_1,\ldots,X_n]/I) + 2$. *Then* $\mu(I) = \mu(I/I^2)$.

COROLLARY 2. *Let X be an irreducible nonsingular affine variety of dimension d over a field k. Let Y be a closed subset in $X \times \mathbf{A}_k^n$ which has pure dimension one. Assume $n \geqslant 2$. Then if the conormal sheaf of Y in $X \times \mathbf{A}_k^n$ is trivial, Y is a complete intersection in $X \times \mathbf{A}_k^n$.*

Corollary 1 is immediate due to a lemma of Bass [**B**, Lemma 3]. Corollary 2 was established in [**BR**, Corollary 3.3].

2. The main theorem. $\mathbf{v} \equiv L(\mathbf{v}) \pmod{E_n(R[X])}$.

The case when R is a local ring was dealt with by Suslin—see [**L**, Chapter III, Lemma 2.8] for instance. For the sake of completeness we include a proof.

(2.1) PROPOSITION (SUSLIN). *Let R be a local ring and $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$, have an entry which is an associate of a monic polynomial. Then $\mathbf{v} \equiv \mathbf{e}_1$ $\pmod{E_n(R[X])}$.*

PROOF. Without any loss of generality, let $\mathbf{v} = (v_1,\ldots,v_n) \in \mathrm{Um}_n(R[X])$, with v_1 an associate of a monic polynomial. Then $A = R[X]/(v_1(X))$ is semilocal, and so $\mathrm{Sl}_{n-1}(A) = E_{n-1}(A)$. Hence, going modulo (v_1) and then lifting, we may transform $\mathbf{v}$ to $(v_1, 1 + v_1v_2', v_1v_3',\ldots,v_1v_n')$ by an elementary action, where $v_i' \in R[X]$ for $2 \leqslant i \leqslant n$. It is now easy to complete the proof.

To reduce to the local case, we are led by ideas of L. N. Vaserstein—see [**L**, Theorem 2.4]—to consider the "local-global" nature of the *action of $E_n(R[X])$*, $n \geqslant 3$, in sending a vector $\mathbf{v} \in \mathrm{Um}_n(R[X])$ to its projection $\mathbf{v}(0)$. For this we shall use the following slight variant of a lemma of Suslin [**S**, Lemma 3.4], which can be proved in an identical fashion.

(2.2) LEMMA (SUSLIN). *Suppose $\alpha \in E_n(R_s[X])$, $\alpha(0) = I_n$, for some $s \in R$. Then there exists a natural number k such that for any $r_1, r_2 \in R[X]$ with $r_1 - r_2 \in s^kR[X]$, $\alpha(r_1X)^{-1}\alpha(r_2X) \in E_n(R[X])$.*

(2.3) LOCAL - GLOBAL THEOREM. *Let* $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$. *Suppose that for all* $\mathfrak{m} \in \mathrm{Max}\, R$, $\mathbf{v} \equiv \mathbf{v}(0) \pmod{E_n(R_{\mathfrak{m}}[X])}$. *Then* $\mathbf{v} \equiv \mathbf{v}(0) \pmod{E_n(R[X])}$.

PROOF. Let $J = \{s \in R \mid \mathbf{v} \equiv \mathbf{v}(0) \pmod{E_n(R_s[X])}\}$ denote the 'Quillen ideal' of $\mathbf{v}$. In view of the assumption, it suffices to show that this set is actually an ideal.

Clearly, one only needs to check that $s_1, s_2 \in J \Rightarrow s_1 + s_2 \in J$. Inverting $s_1 + s_2$, we may assume $s_1 + s_2 = 1$.

Let $\sigma_i \in E_n(R_{s_i}[X])$, $i = 1, 2$, with $\sigma_i \mathbf{v} = \mathbf{v}(0)$. We may assume $\sigma_i(0) = I_n$, $i = 1, 2$. Let $\alpha = \sigma_2\sigma_1^{-1} \in E_n(R_{s_1 s_2}[X])$. By (2.2) there exists $k > 0$ such that $\alpha(aX) \in E_n(R_{s_2}[X])$ if $s_1^k | a$, and $\alpha(aX)^{-1}\alpha(X) \in E_n(R_{s_1}[X])$ if $s_2^k|(1 - a)$. Since $(s_1, s_2) = 1$, given any integer k, such an a is readily available.

Observe that, since $\mathbf{v}(0)$ is a constant vector and $X \to aX$ is R-linear, $\alpha(X)\alpha(aX)^{-1}$ preseves $\mathbf{v}(0)$! But then

$$\sigma_2\sigma_1^{-1} = \alpha(X) = \alpha(aX)\big(\alpha(aX)^{-1}\alpha(X)\big) = \beta_2\beta_1$$

with $\beta_i \in E_n(R_{s_i}[X])$, $i = 1, 2$, and $\beta_i\mathbf{v}(0) = \mathbf{v}(0)$. Replacing σ_2 by $\beta_2^{-1}\sigma_2$, σ_1 by $\beta_1\sigma_1$, we obtain a $\gamma \in \mathrm{Sl}_n(R[X])$ with $\gamma\mathbf{v} = \mathbf{v}(0)$, and $\gamma_{s_i} = \sigma_i$, $i = 1, 2$. Since γ is "locally" elementary, by **[S**, Theorem 3.1], $\gamma \in E_n(R[X])$. Thus, J is an ideal.

A Horrock's-like argument now completes the proof of the main theorem.

(2.4) Theorem. *Let* $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$, *be a special unimodular vector. Then* $\mathbf{v} \equiv L(\mathbf{v}) \pmod{E_n(R[X])}$.

Proof. Regard $\mathbf{v} = (v_1, \dots, v_n) \in \mathrm{Um}_n(R[X - 1])$! By (2.1), $\mathbf{v}_{\mathfrak{m}} \equiv \mathbf{v}(1) \pmod{E_n(R_{\mathfrak{m}}[X - 1])}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$ since $\mathbf{v}$ is special. Therefore, by the Local-Global Theorem (2.3), $\mathbf{v} \equiv \mathbf{v}(1) \pmod{E_n(R[X - 1])}$. It thus suffices to show that $\mathbf{v}(1) \equiv L(\mathbf{v}) \pmod{E_n(R)}$.

Let $m_i = \deg v_i$, and put $w_i = X^{-m_i}v_i \in R[X^{-1}]$, for $1 \leqslant i \leqslant n$. Then $\mathbf{w} = (w_1, \dots, w_n) \in \mathrm{Um}_n(R[X^{-1}])$, as $\mathbf{w}$ is unimodular after inverting X^{-1}, and $\mathbf{w}(0) = L(\mathbf{v}) \in \mathrm{Um}_n(R)$. Applying (2.1) and (2.3) as above we get

$$\mathbf{w} \equiv \mathbf{w}(0) = L(\mathbf{v}) \pmod{E_n(R[X^{-1}])}.$$

Putting $X^{-1} = 1$, we get

$$\mathbf{w}(1) = \mathbf{v}(1) \equiv L(\mathbf{v}) \pmod{E_n(R)}!$$

(2.5) Corollary. *Let* $\mathbf{v} \in \mathrm{Um}_n(R[X])$, $n \geqslant 3$, *with some entry of* $\mathbf{v}$ *an associate of a monic polynomial. Then* $\mathbf{v}$ *can be completed to an elementary matrix* $\alpha \in E_n(R[X])$.

(2.6) Remark. We indicate another interesting proof of (2.4) which arose out of trying to understand why the proof of the theorem in **[M]** works.

Think of $\mathbf{v} \in \mathrm{Um}_n(R[X, T, T^{-1}])$!. Consider the R-linear automorphism φ of $R[X, T, T^{-1}]$ which fixes T and sends X to $X - T + T^{-1}$. Clearly, it suffices to show $\varphi(\mathbf{v}) \equiv L(\mathbf{v}) \pmod{E_n(R[X, T, T^{-1}])}$. The effect of φ on a polynomial $f \in R[X]$ is to change it to a Laurent polynomial in T whose leading and lowest coefficient $= L(f)$ *up to a sign*. If $\varphi(\mathbf{v}) = (\varphi(v_1), \dots, \varphi(v_n))$ and m_i are least integers so chosen that $T^{m_i}\varphi(v_i) = w_i \in R[X, T]$, then $\mathbf{w} = (w_1, \dots, w_n) \in \mathrm{Um}_n(R[X, T])$. Applying (2.1) and (2.3),

$$\mathbf{w} \equiv \mathbf{w}(0) \pmod{E_n(R[X, T])} \quad \text{and} \quad \mathbf{w}(0) \equiv L(\mathbf{v}) \pmod{E_n(R)}.$$

But then $\varphi(\mathbf{v}) \equiv L(\mathbf{v}) \pmod{E_n(R[X, T])}$. Hence $\mathbf{v} \equiv L(\mathbf{v}) \pmod{E_n(R[X])}$.

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