AN ELEMENTARY TRANSFORMATION OF A SPECIAL UNIMODULAR VECTOR TO ITS TOP COEFFICIENT VECTOR

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Abstract. Let \( R \) be a commutative ring, \( v(X) \) a unimodular \( n \)-vector \((n \geq 3)\) over \( R[X] \). Suppose the leading coefficients in \( v(X) \) form a unimodular vector \( L(v) \) over \( R \). Then some element in \( E_n(\mathbb{R}[X]) \) will transform \( v(X) \) to \( L(v) \).

1. Introduction and motivation. For a commutative ring \( R \), \( E_n(R) \) will denote the subgroup of the determinant one matrices \( \text{SL}_n(R) \) generated by \( E_{ij}(\lambda) = I_n + \lambda e_{ij}, \) where \( I_n \) denotes the \( n \times n \) identity matrix, \( 1 \leq i \neq j \leq n, \lambda \in R, \) and \( e_{ij} \) is the matrix whose \((i, j)\)th entry is 1 and all other entries are zeros. If \( \alpha \in E_n(R) \), call it an elementary matrix.

A vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) is called unimodular if there is a vector \( w = (w_1, \ldots, w_n) \in \mathbb{R}^n \) with \( v \cdot w^t = \sum_i v_i w_i = 1. \) \( \text{Um}_n(R) \) will denote the set of all unimodular vectors in \( \mathbb{R}^n. \) Clearly, \( E_n(R) \) acts on \( \text{Um}_n(R). \) Denote by \( \equiv (\mod E_n(R)) \) vectors in the same orbit under this action.

For a polynomial \( f \in \mathbb{R}[X], \) \( L(f) \) will denote the leading coefficient of \( f. \)

A vector \( v = (v_1, \ldots, v_n) \in \text{Um}_n(\mathbb{R}[X]) \) is called a special unimodular vector if \( L(v) = (L(v_1), \ldots, L(v_n)) \in \text{Um}_n(R). \)

The principal aim of this short note is to point out

**Theorem.** Let \( v \in \text{Um}_n(\mathbb{R}[X]), n \geq 3, \) be a special unimodular vector. Then \( v \equiv L(v) (\mod E_n(\mathbb{R}[X])). \)

This simplifies and extends a result of A. A. Suslin in [S, Proposition 2.4].

The theorem settles affirmatively a question raised by M. P. Murthy during discussions, viz. if \( v \in \text{Um}_n(\mathbb{R}[X]), n \geq 3, \) has a monic polynomial as one of its entries, then is \( v = e_1 = (1, 0, \ldots, 0) (\mod E_n(\mathbb{R}[X]))? \)

Murthy's question was raised in relation to the following problem on the efficient generation of ideals in polynomial rings.

**Problem.** Let \( R \) be a noetherian ring, and \( I \) an ideal in the polynomial ring \( \mathbb{R}[X] \) containing a monic polynomial. Let \( I \) satisfy the following condition on its conormal bundle; viz the \( R/I \)-module \( I/I^2: \)

\[
\mu_{R/I}(I/I^2) = \text{the least number of generators of } I/I^2 \text{ as a } R/I\text{-module} \\
\geq \dim(\mathbb{R}[X]/I) + 2.
\]

Then is \( \mu(I) = \mu(I/I^2)? \)
Recently, S. Mandal [M] settled the above problem affirmatively. Earlier, N. Mohan Kumar's reductions in [MK] had established a surjection \( P \to I \to 0 \) with \( P \) a projective \( R[X] \)-module of rank \( \mu(I/I^2) \). By utilizing the finer information available now by our theorem, we can considerably simplify the treatment in [MK] and show that \( I \) is the onto image of a free module of rank \( \mu(I/I^2) \), thereby providing a quick alternative solution to the above problem.

Let me mention two applications of this problem.

**Corollary 1.** Let \( R \) be a noetherian ring, and \( I \) an ideal in \( R[X_1, \ldots, X_n] \), with height \( I > \text{dim } R \), and \( \mu(I/I^2) \geq \text{dim}(R[X_1, \ldots, X_n]/I) + 2 \). Then \( \mu(I) = \mu(I/I^2) \).

**Corollary 2.** Let \( X \) be an irreducible nonsingular affine variety of dimension \( d \) over a field \( k \). Let \( Y \) be a closed subset in \( X \times A^n_k \) which has pure dimension one. Assume \( n \geq 2 \). Then if the conormal sheaf of \( Y \) in \( X \times A^n_k \) is trivial, \( Y \) is a complete intersection in \( X \times A^n_k \).

Corollary 1 is immediate due to a lemma of Bass [B, Lemma 3]. Corollary 2 was established in [BR, Corollary 3.3].

2. The main theorem. \( v \equiv L(v) \pmod{E_n(R[X])} \).

The case when \( R \) is a local ring was dealt with by Suslin—see [L, Chapter III, Lemma 2.8] for instance. For the sake of completeness we include a proof.

(2.1) **Proposition (Suslin).** Let \( R \) be a local ring and \( v \in \text{Um}_n(R[X]), n \geq 3 \), have an entry which is an associate of a monic polynomial. Then \( v \equiv e_1 \pmod{E_n(R[X])} \).

**Proof.** Without any loss of generality, let \( v = (v_1, \ldots, v_n) \in \text{Um}_n(R[X]), \) with \( v_1 \) an associate of a monic polynomial. Then \( A = R[X]/(v_1(X)) \) is semilocal, and so \( S_{n-1}(A) = E_{n-1}(A) \). Hence, going modulo \((v_1)\) and then lifting, we may transform \( v \) to \((v_1, 1 + v_1v'_2, v_1v'_3, \ldots, v_1v'_n)\) by an elementary action, where \( v'_i \in R[X] \) for \( 2 \leq i \leq n \). It is now easy to complete the proof.

To reduce to the local case, we are led by ideas of L. N. Vaserstein—see [L, Theorem 2.4]—to consider the “local-global” nature of the action of \( E_n(R[X]) \), \( n \geq 3 \), in sending a vector \( v \in \text{Um}_n(R[X]) \) to its projection \( v(0) \). For this we shall use the following slight variant of a lemma of Suslin [S, Lemma 3.4], which can be proved in an identical fashion.

(2.2) **Lemma (Suslin).** Suppose \( \alpha \in E_n(R_m[X]), \) \( \alpha(0) = I_n \), for some \( s \in R \). Then there exists a natural number \( k \) such that for any \( r_1, r_2 \in R[X] \) with \( r_1 - r_2 \in s^kR[X], \) \( \alpha(r_1X) = \alpha(r_2X) \in E_n(R[X]) \).

(2.3) **Local-Global Theorem.** Let \( v \in \text{Um}_n(R[X]), n \geq 3 \). Suppose that for all \( m \in \text{Max } R, v \equiv v(0) \pmod{E_n(R_m[X])} \). Then \( v \equiv v(0) \pmod{E_n(R[X])} \).

**Proof.** Let \( J = \{ s \in R \mid v \equiv v(0) \pmod{E_n(R_m[X]))} \} \) denote the ‘Quillen ideal’ of \( v \). In view of the assumption, it suffices to show that this set is actually an ideal.
Clearly, one only needs to check that \( s_1, s_2 \in J \Rightarrow s_1 + s_2 \in J \). Inverting \( s_1 + s_2 \), we may assume \( s_1 + s_2 = 1 \).

Let \( \sigma_i \in E_n(R_{s_i}(X)), i = 1, 2 \), with \( \sigma_1 v = v(0) \). We may assume \( \sigma_i(0) = I_n, i = 1, 2 \). Let \( \alpha = \sigma_2 \sigma_1^{-1} \in E_n(R_{s_1 s_2}(X)) \). By (2.2) there exists \( k > 0 \) such that \( \alpha(aX) \in E_n(R_{s_1}(X)) \) if \( S_1^k a \), and \( \alpha(aX)^{-1} \alpha(X) \in E_n(R_{s_1}(X)) \) if \( S_2^{-k}(1 - a) \). Since \( (s_1, s_2) = 1 \), given any integer \( k \), such an \( a \) is readily available.

Observe that, since \( v(0) \) is a constant vector and \( X \to aX \) is \( R \)-linear, \( \alpha(X) \alpha(aX)^{-1} \) preserves \( v(0) \! \). But then

\[
\sigma_2 \sigma_1^{-1} = \alpha(X) = \alpha(aX) \left( \alpha(aX)^{-1} \alpha(X) \right) = \beta_2 \beta_1
\]

with \( \beta_i \in E_n(R_{s_i}(X)), i = 1, 2 \), and \( \beta_1 v(0) = v(0) \). Replacing \( \sigma_1 \) by \( \beta_1^{-1} \sigma_1 \), \( \sigma_1 \) by \( \beta_1 \sigma_1 \), we obtain a \( \gamma \in \text{SL}_n(R[X]) \) with \( \gamma v = v(0) \), and \( \gamma_{s_i} = \sigma_i, i = 1, 2 \). Since \( \gamma \) is \"locally\" elementary, by [S, Theorem 3.1], \( \gamma \in E_n(R[X]) \). Thus, \( J \) is an ideal.

A Horrock’s-like argument now completes the proof of the main theorem.

(2.4) Theorem. Let \( v \in \text{Um}_n(R[X]), n \geq 3 \), be a special unimodular vector. Then \( v \equiv L(v) \pmod{E_n(R[X])} \).

Proof. Regard \( v = (v_1, \ldots, v_n) \in \text{Um}_n(R[X - 1]) \)! By (2.1), \( v_m \equiv v(1) \pmod{E_n(R_m[X - 1])} \) for all \( m \in \text{Max}(R) \) since \( v \) is special. Therefore, by the Local-Global Theorem (2.3), \( v \equiv v(1) \pmod{E_n(R[X - 1])} \). It thus suffices to show that \( v(1) \equiv L(v) \pmod{E_n(R)} \).

Let \( m_i = \deg v_i \), and put \( w_i = X^{-m_i} v_i \in R[X^{-1}] \), for \( 1 \leq i \leq n \). Then \( w = (w_1, \ldots, w_n) \in \text{Um}_n(R[X^{-1}]) \), as \( w \) is unimodular after inverting \( X^{-1} \), and \( w(0) = L(v) \equiv \text{Um}_n(R) \). Applying (2.1) and (2.3) as above we get

\[
w \equiv w(0) = L(v) \pmod{E_n(R[X^{-1}])}.
\]

Putting \( X^{-1} = 1 \), we get

\[
w(1) = v(1) \equiv L(v) \pmod{E_n(R)}!
\]

(2.5) Corollary. Let \( v \in \text{Um}_n(R[X]), n \geq 3 \), with some entry of \( v \) an associate of a monic polynomial. Then \( v \) can be completed to an elementary matrix \( \alpha \in E_n(R[X]) \).

(2.6) Remark. We indicate another interesting proof of (2.4) which arose out of trying to understand why the proof of the theorem in [M] works.

Think of \( v \in \text{Um}_n(R[X, T, T^{-1}]) \). Consider the \( R \)-linear automorphism \( \varphi \) of \( R[X, T, T^{-1}] \) which fixes \( T \) and sends \( X \) to \( X - T + T^{-1} \). Clearly, it suffices to show \( \varphi(v) \equiv L(v) \pmod{E_n(R[X, T, T^{-1}])} \). The effect of \( \varphi \) on a polynomial \( f \in R[X] \) is to change it to a Laurent polynomial in \( T \) whose leading and lowest coefficient = \( L(f) \) up to a sign. If \( \varphi(v) = (\varphi(v_1), \ldots, \varphi(v_n)) \) and \( m_i \) are least integers so chosen that \( T^{-m_i} \varphi(v_i) = w_i \in R[X, T] \), then \( w = (w_1, \ldots, w_n) \in \text{Um}_n(R[X, T]) \). Applying (2.1) and (2.3),

\[
w \equiv w(0) \pmod{E_n(R[X, T])} \quad \text{and} \quad w(0) \equiv L(v) \pmod{E_n(R)}.
\]

But then \( \varphi(v) = L(v) \pmod{E_n(R[X, T])} \). Hence \( v = L(v) \pmod{E_n(R[X])} \).
REFERENCES


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