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MARKOV'S INEQUALITY FOR POLYNOMIALS WITH REAL ZEROS

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ABSTRACT. Markov's inequality asserts that $\|p_n'\| \leqslant n^2\|p_n\|$ for any polynomial p_n of degree n. (We denote the supremum norm on $[-1,1]$ by $\|\cdot\|$.) In the case that p_n has all real roots, none of which lie in $[-1,1]$, Erdös has shown that $\|p_n'\| \leqslant en\|p_n\|/2$. We show that if p_n has $n-k$ real roots, none of which lie in $[-1,1]$, then $\|p_n'\| \leqslant cn(k+1)\|p_n\|$, where c is independent of n and k. This extension of Markov's and Erdös' inequalities was conjectured by Szabados.

Introduction. Markov's inequality asserts that

$$\|p_n'\|_{[-1,1]} \leqslant n^2\|p_n\|_{[-1,1]} \tag{1}$$

for any poynomial $p_n \in \pi_n$ **[2** and **3]**. (π_n denotes the algebraic polynomials of degree at most n and $\|\cdot\|_A$ denotes the supremum norm on A.) Erdös **[1]** in 1940 offered the following refinement of Markov's inequality. If $p_n \in \pi_n$ and p_n has all its roots in $\mathbf{R} - (-1,1)$, then

$$\|p_n'\|_{[-1,1]} \leqslant \frac{en}{2}\|p_n\|_{[-1,1]}. \tag{2}$$

Inequality (1) iterates to give bounds for the kth derivative of a polynomial. However, we cannot proceed inductively with inequality (2) since some of the roots of the derivatives may be in $[-1,1]$. With this in mind, Szabados and Varma established a version of (2) for polynomials of degree n with all real roots and at most one root in $[-1,1]$, namely, for such a polynomial p_n,

$$\|p_n'\|_{[-1,1]} \leqslant c_1 n\|p_n\|_{[-1,1]}, \tag{3}$$

where c_1 is independent of n **[5]**. This, of course, yields the following inequality:

$$\|p_n''\|_{[-1,1]} \leqslant c_2 n^2\|p_n\|_{[-1,1]} \tag{4}$$

for any $p_n \in \pi_n$ that has all its roots in $\mathbf{R} - (-1,1)$. In **[6]** Szabados proposed the following

Conjecture. If p_n is a polynomial of degree n and p_n has at least $n-k$ roots in $\mathbf{R} - (-1,1)$, then there is a constant c ($c \leqslant 9$) so that

$$\|p_n'\|_{[-1,1]} \leqslant cn(k+1)\|p_n\|_{[-1,1]}. \tag{5}$$

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It is our intention to prove this slightly strengthened form of Szabados' conjecture. In its original form the conjecture had the additional assumption that all the roots of p_n be real. Up to the constant this result is best possible; Szabados in **[6]** constructs polynomials p_n of degree n with $n-k$ roots in $\mathbf{R}-(-1,1)$ so that

$$\|p_n'\|_{[-1,1]} \geqslant \frac{n\cdot k}{2}\|p_n\|_{[-1,1]} \qquad (0<k\leqslant n).$$

It is apparent from (1) and (2) that the best constant must depend on k. Some related results may be found in **[4]**.

Inequalities for the higher derivatives of polynomials with real roots can now be derived straightforwardly from (5). For example,

THEOREM. *If $p_n \in \pi_n$ has at least $n-k$ zeros in $\mathbf{R}-(-1,1)$, then*

$$\|p_n^{(m)}\|_{[-1,1]} \leqslant c_m \frac{n!(k+m)!}{(n-m)!k!}\|p_n\|_{[-1,1]},$$

where $c_m \leqslant 9^m$ depends only on m.

2. Proof of the Conjecture. Let c_{2k} be the $2k$th Chebychev polynomial shifted to the interval $[0,2]$ and normalized to have lead coefficient 1. Let $\alpha_1<\alpha_2<\cdots<\alpha_k$ be the roots of c_{2k} in $[1,2]$ and let

$$t_k := \prod_{i=1}^{k}(x-\alpha_i). \tag{6}$$

LEMMA 1. *The polynomial $q_k := (x+2m/k)^{m+k}t_k(x)$ has the following property. If $\alpha_0=0$ and $\alpha_{n+1}=1$, then, for $i=1,2,\dots,n$,*

$$\left\|\left(x+\frac{2m}{k}\right)^{m+k}t_k\right\|_{[\alpha_{i-1},\alpha_1]} > \left\|\left(x+\frac{2m}{k}\right)^{m+k}t_k\right\|_{[\alpha_i,\alpha_{i+1}]}, \tag{7}$$

where the maximums on successive intervals occur with alternating sign.

PROOF. Let $0<\beta_1<\cdots<\beta_k$ be the roots of c_{2k} in $[0,1]$. We observe that $x/(x-\beta_i)$ is positive and decreasing on $(\beta_k,\infty\,]$ and that c_{2k} equioscillates on the intervals in question (i.e. c_{2k} satisfies (7) with equality). We now note that

$$x^k t_k = \left(\prod_{i=1}^{k}\frac{x}{(x-\beta_i)}\right)c_{2k}$$

satisfies the conclusion of Lemma 1. To finish the proof we need only observe that $(x+2m/k)^{m+k}/x^k$ is decreasing on $[0,2]$. □

Let $n=2k+m$ and let

$$s_n(x) := \frac{1}{(1+m/k)^n}q_k\left(\left(1+\frac{m}{k}\right)x+\left(1-\frac{m}{k}\right)\right).$$

(We have shifted from $[-2m/k,2]$ to $[-1,1]$.) This polynomial will act as a kind of near extremal polynomial for the Conjecture. Let $\gamma_1<\gamma_2<\cdots<\gamma_k$ be the roots of s_n in $(-1,1)$. We collect the properties of s_n that we require in the next lemma.

LEMMA 2. *For $n = m + 2k$ and s_n as above*:
(a) $s_n(x) = (x+1)^{m+k}\prod_{i=1}^{k}(x - \gamma_i)$,
(b) $\sum_{i=1}^{k}(1/(1-\gamma_i)) \leqslant 4k(n-k)$, *and*
(c) *for* $i = 1,\ldots,k$, $\gamma_0 = -1$ *and* $\gamma_{k+1} = 1$

$$\|s_n\|_{[\gamma_{i-1},\gamma_i]} \geqslant \|s_n\|_{[\gamma_i,\gamma_{i+1}]}.$$

PROOF. Parts (a) and (c) are immediate from the construction of s_n. Part (b) follows from the observation that, for α_i as in (6).

$$\sum_{i=1}^{k} \frac{1}{2-\alpha_i} \leqslant \frac{c_{2k}'(2)}{c_{2k}(2)} = 4k^2$$

and the observation that

$$1 - \gamma_i = \frac{1}{1+m/k}(2-\alpha_i). \quad \square$$

Let $p_n^* \in \pi_n$ maximize

(8) $$|p_n'(1)|/\|p_n\|_{[-1,1]},$$

where the maximum is taken over all polynomials in π_n that have all but at most k roots in $\mathbf{R} - (-1,1)$. The information we need about p_n^* is contained in the next lemma.

LEMMA 3. *Let p_n^* be as above. Then*
(a) *p_n^* has k simple roots $\delta_1 < \cdots < \delta_k$ in* $(-1,1)$, *p_n^* has $n-k$ roots at* ± 1, *and p_n^* achieves its maximum modulus on each of the intervals* $[-1,\delta_1], [\delta_1,\delta_2],\ldots,[\delta_k,1]$.
(b) *Either p_n^* has no roots in* $[-1,\infty)$ *or p_n^* has exactly one root at* 1.

PROOF. The proof of (a) is a simple and standard perturbation argument (if p_n^* did not satisfy (a) then it would be possible to perturb p_n^* to reduce its norm on $[-1,1]$ without decreasing the derivative at 1). We will prove only that p_n has no roots in $(1,\infty)$, the other parts are similar. First suppose that p_n^* has two roots at $\alpha > 1$ and $\beta > 1$. Consider

$$v_n(x) := \frac{p_n^*(x)(x-1)^2}{(x-\alpha)(x-\beta)}.$$

Then for sufficiently small $\varepsilon > 0$:
(i) $\|p_n^* - \varepsilon v_n\|_{[-1,1]} < \|p_n^*\|_{[-1,1]}$,
(ii) $|(p_n^* - \varepsilon v_n)'(1)| = |p_n^{*\prime}(1)|$,
(iii) *$p_n^* - \varepsilon v_n$ has all but at most k roots in* $\mathbf{R} - (1,1)$.

Part (iii) follows since $(x-\alpha)(x-\beta) - \varepsilon(x-1)^2$ has two roots in $[1,\infty]$ for sufficiently small ε. (Note α may equal β.) However, this contradicts the maximality of p_n^*.

Next we suppose that p_n^* has exactly one (nonrepeat) root at $\alpha > 1$. Now we argue as before by considering

$$v_n(x) := p_n^*(x)\frac{(x-1)^2}{(x-1)(x-\alpha)}.$$

If $p_n^*(1) \neq 0$ we must observe that in this case $\operatorname{sign}(p_n^*(1)) = \operatorname{sign}(p_n^{*\prime}(1))$ and, hence, that

$$v_n'(1) = \frac{(1-\alpha)p_n^*(1)}{(1-\alpha)^2}$$

has the opposite sign to $p_n^{*\prime}(1)$. The last observation requires noticing that if $p_n^{*\prime}(1)$ has opposite sign to $p_n^*(1)$, then $p_n^{*\prime}$ has all its zeros in $(-\infty, 1]$ and, hence, $p_n''(1) \neq 0$. Thus, $|p_n^{*\prime}|$ is increasing on $[1, \infty)$, $|p_n|$ is decreasing on $[1, \alpha)$ and $p_n^*(x + (\alpha - 1))$ violates the maximality assumptions on $[-1, 1]$.

Part (b) follows since if p_n^* has two or more zeros at 1, then $p_n'(1)$ would also equal zero. □

LEMMA 4. *If $p_n \in \pi_n$ has at least $(n-k)$ roots in* $\mathbf{R} - (-1,1)$, *then*

$$|p_n'(1)| \leqslant \tfrac{9}{2}(k+1)n\|p_n\|_{[-1,1]}.$$

PROOF. If $2k \geqslant n$, then the lemma follows from Markov's inequality, so we may suppose $2k < n$. Suppose there exists p_n, as above, so that

$$|p_n'(1)| > \tfrac{9}{2}(k+1)(n-k)\|p_n\|_{[-1,1]},$$

and let q_n be the maximal such p_n. By Lemma 3, this q_n equioscillates $k+1$ times on $[-1, 1]$.

We shall first consider the case where q_n has no root at 1.

The key to the proof is to observe that the roots of q_n lie to the left of the roots of s_n (as defined in Lemma 3). We may write

$$q_n(x) = (x+1)^{n-k}\prod_{i=1}^{k}(x-\rho_i),$$

where $-1 < \rho_1 < \cdots < \rho_k < 1$. Also,

$$s_n(x) = (x+1)^{n-k}\prod_{i=1}^{k}(x-\gamma_i).$$

The claim is that $\gamma_i \geqslant \rho_i$ for each i. This is seen as follows.Choose the largest i for which $\rho_i > \gamma_i$. Then pick η so that $\|\eta q_n\|_{[\gamma_i,1]} = \|s_n\|_{[\gamma_i,1]}$. (We will specify the sign of η later.)We can deduce from the equioscillation of q_n that $\eta q_n - s_n$ has at least $k - i$ roots on $[\beta, 1]$, where β is the first point greater than ρ_i where ηq_n achieves its maximum modulus. From Lemma 2(c) we deduce that $\eta q_n - s_n$ has at least $i - 1$ roots on $(-1, \alpha)$, where α is the largest point less than γ_i where s_n achieves its maximum modulus. We need only observe that if we choose the sign of η so that

$$\operatorname{sign} \eta q_n(\beta) = -\operatorname{sign} s_n(\alpha),$$

then $\eta q_n - s_n$ must have 2 roots in (α, β). Thus, $\eta q_n - s_n$ has $n+1$ roots which is a contradiction and we conclude that $\rho_i \leqslant \gamma_i$.

We now observe that, since $\rho_i \leqslant \gamma_i < 1$,

$$\frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}} = \frac{q_n'(1)}{q_n(1)} = \sum_{i=1}^{k} \frac{1}{1-\rho_i} + \sum_{i=1}^{n-k} \frac{1}{1-(-1)}$$

$$\leqslant \sum_{i=1}^{k} \frac{1}{1-\gamma_i} + \frac{n-k}{2} \leqslant 4k(n-k) + \frac{n-k}{2},$$

where the later inequality follows from Lemma 2. This is a contradiction.

In the case where q_n has exactly one root at 1 we proceed as follows. Let $d > 1$ be the unique point in $(1, \infty)$, where $|q_n(d)| = \|q_n\|_{[-1,1]}$. We can now consider q_n on $[-1, d]$. We note that

$$\frac{|q_n'(d)|}{\|q_n\|_{[-1,1]}} \geqslant \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}$$

since $|q_n'|$ is increasing on $[1, \infty)$. We can repeat verbatim the argument of the first part applied to

$$\tilde{q}(x) = q_n\left(x\left(\frac{d+1}{2}\right) + \left(\frac{d-1}{2}\right)\right)$$

with k replaced by $k+1$. This allows us to deduce the contradiction that

$$\frac{|q'(1)|}{\|q_n\|_{[-1,1]}} \leqslant \frac{|\tilde{q}_n(1)|}{\|\tilde{q}_n\|_{[-1,1]}} \leqslant 4(k+1)(n-k) + \frac{n-k}{2}. \quad \square$$

The proof of the Conjecture is now straightforward.

PROOF OF CONJECTURE. Let p_n be a polynomial of degree n with $n-k$ roots in $\mathbf{R} - (-1,1)$. Let x_0 be a point in $[-1,1]$, where p_n' achieves its maximum modulus. We suppose $x_0 \leqslant 0$ ($x_0 > 0$ follows analogously). Let $ax + b$ map $[x_0, 1]$ one-to-one onto $[-1,1]$ in such a way that $x_0 \to 1$.

Note that $|a| < 2$. Thus, if $v_n(ax+b) = p_n(x)$, then

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[-1,1]}} \leqslant \frac{2|v_n'(1)|}{\|v_n\|_{[-1,1]}} = 9n(k+1),$$

where the last inequality follows from Lemma 4 and the observation that v_n has at least as many roots as p_n in $\mathbf{R} - [-1,1]$. $\square$

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