MARKOV'S INEQUALITY FOR POLYNOMIALS WITH REAL ZEROS

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Abstract. Markov's inequality asserts that \( \| p_n \| \leq n^2 \| p_n \| \) for any polynomial \( p_n \) of degree \( n \). (We denote the supremum norm on \([-1, 1]\) by \( \| \cdot \| \).) In the case that \( p_n \) has all real roots, none of which lie in \([-1, 1]\), Erdős has shown that \( \| p'_n \| \leq en \| p_n \| /2 \). We show that if \( p_n \) has \( n - k \) real roots, none of which lie in \([-1, 1]\), then \( \| p'_n \| \leq cn(k + 1) \| p_n \| \), where \( c \) is independent of \( n \) and \( k \). This extension of Markov's and Erdős' inequalities was conjectured by Szabados.

Introduction. Markov’s inequality asserts that

\[
\| p_n \|_{[-1,1]} \leq n^2 \| p_n \|_{[-1,1]}
\]

for any polynomial \( p_n \in \pi_n \) [2 and 3]. (\( \pi_n \) denotes the algebraic polynomials of degree at most \( n \) and \( \| \cdot \|_A \) denotes the supremum norm on \( A \).) Erdős [1] in 1940 offered the following refinement of Markov’s inequality. If \( p_n \in \pi_n \) and \( p_n \) has all its roots in \( \mathbb{R} - (-1,1) \), then

\[
\| p'_n \|_{[-1,1]} \leq \frac{en}{2} \| p_n \|_{[-1,1]}
\]

Inequality (1) iterates to give bounds for the \( k \)th derivative of a polynomial. However, we cannot proceed inductively with inequality (2) since some of the roots of the derivatives may be in \([-1, 1]\). With this in mind, Szabados and Varma established a version of (2) for polynomials of degree \( n \) with all real roots and at most one root in \([-1, 1] \), namely, for such a polynomial \( p_n \),

\[
\| p'_n \|_{[-1,1]} \leq c_1 n \| p_n \|_{[-1,1]},
\]

where \( c_1 \) is independent of \( n \) [5]. This, of course, yields the following inequality:

\[
\| p''_n \|_{[-1,1]} \leq c_2 n^2 \| p_n \|_{[-1,1]}
\]

for any \( p_n \in \pi_n \) that has all its roots in \( \mathbb{R} - (-1,1) \). In [6] Szabados proposed the following

Conjecture. If \( p_n \) is a polynomial of degree \( n \) and \( p_n \) has at least \( n - k \) roots in \( \mathbb{R} - (-1,1) \), then there is a constant \( c \) (\( c \leq 9 \)) so that

\[
\| p'_n \|_{[-1,1]} \leq cn(k + 1) \| p_n \|_{[-1,1]}
\]
It is our intention to prove this slightly strengthened form of Szabados' conjecture. In its original form the conjecture had the additional assumption that all the roots of $p_n$ be real. Up to the constant this result is best possible; Szabados in [6] constructs polynomials $p_n$ of degree $n$ with $n - k$ roots in $\mathbb{R} - (-1, 1)$ so that

$$\|p_n\|_{[-1, 1]} \geq \frac{n \cdot k}{2} \|p_n\|_{[-1, 1]} \quad (0 < k \leq n).$$

It is apparent from (1) and (2) that the best constant must depend on $k$. Some related results may be found in [4].

Inequalities for the higher derivatives of polynomials with real roots can now be derived straightforwardly from (5). For example,

**Theorem.** If $p_n \in \pi_n$ has at least $n - k$ zeros in $\mathbb{R} - (-1, 1)$, then

$$\|p_n^{(m)}\|_{[-1, 1]} \leq c_m n! (n - m)! \|p_n\|_{[-1, 1]},$$

where $c_m \leq 9^m$ depends only on $m$.

2. **Proof of the Conjecture.** Let $c_{2k}$ be the $2k$th Chebychev polynomial shifted to the interval $[0, 2]$ and normalized to have lead coefficient 1. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be the roots of $c_{2k}$ in $[1, 2]$ and let

$$t_k := \prod_{i=1}^{k} (x - \alpha_i).$$

**Lemma 1.** The polynomial $q_k := (x + 2m/k)^{m+k} t_k(x)$ has the following property. If $\alpha_0 = 0$ and $\alpha_{n+1} = 1$, then, for $i = 1, 2, \ldots, n$,

$$\| (x + \frac{2m}{k})^{m+k} t_k \|_{[\alpha_{i-1}, \alpha_i]} > \| (x + \frac{2m}{k})^{m+k} t_k \|_{[\alpha_i, \alpha_{i+1}]},$$

where the maximums on successive intervals occur with alternating sign.

**Proof.** Let $0 < \beta_1 < \cdots < \beta_k$ be the roots of $c_{2k}$ in $[0, 1]$. We observe that $x/(x - \beta_i)$ is positive and decreasing on $(\beta_k, \infty)$ and that $c_{2k}$ equioscillates on the intervals in question (i.e. $c_{2k}$ satisfies (7) with equality). We now note that

$$x^k t_k = \left( \prod_{i=1}^{k} \frac{x}{(x - \beta_i)} \right) c_{2k}$$

satisfies the conclusion of Lemma 1. To finish the proof we need only observe that $(x + 2m/k)^{m+k}/x^k$ is decreasing on $[0, 2]$. □

Let $n = 2k + m$ and let

$$s_n(x) := \frac{1}{(1 + m/k)^n} q_k \left( \left( 1 + \frac{m}{k} \right) x + \left( 1 - \frac{m}{k} \right) \right).$$

(We have shifted from $[-2m/k, 2]$ to $[-1, 1]$.) This polynomial will act as a kind of near extremal polynomial for the Conjecture. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ be the roots of $s_n$ in $(-1, 1)$. We collect the properties of $s_n$ that we require in the next lemma.
Lemma 2. For \( n = m + 2k \) and \( s_n \) as above:
(a) \( s_n(x) = (x + 1)^{m+k} \prod_{i=1}^{k} (x - \gamma_i) \),
(b) \( \sum_{i=1}^{k} \left( \frac{1}{1 - \gamma_i} \right) \leq 4k(n - k) \), and
(c) for \( i = 1, \ldots, k \), \( \gamma_0 = -1 \) and \( \gamma_{k+1} = 1 \).

Proof. Parts (a) and (c) are immediate from the construction of \( s_n \). Part (b) follows from the observation that, for \( \alpha_i \) as in (6),
\[
\sum_{i=1}^{k} \frac{1}{2 - \alpha_i} \leq \frac{c_{2k}^2(2)}{c_{2k}} = 4k^2
\]
and the observation that
\[
1 - \gamma_i = \frac{1}{1 + m/k} (2 - \alpha_i).
\]

Let \( p^*_n \in \sigma_n \) maximize
\[
|p^*_n(1)|/\|p_n\|_{[-1,1]},
\]
where the maximum is taken over all polynomials in \( \sigma_n \) that have all but at most \( k \) roots in \( \mathbb{R} - (-1,1) \). The information we need about \( p^*_n \) is contained in the next lemma.

Lemma 3. Let \( p^*_n \) be as above. Then
(a) \( p^*_n \) has \( k \) simple roots \( \delta_1 < \cdots < \delta_k \) in \( (-1,1) \), \( p^*_n \) has \( n - k \) roots at \( \pm 1 \), and \( p^*_n \) achieves its maximum modulus on each of the intervals \( [-1, \delta_1], [\delta_1, \delta_2], \ldots, [\delta_k, 1] \).
(b) Either \( p^*_n \) has no roots in \( [-1, \infty) \) or \( p^*_n \) has exactly one root at \( 1 \).

Proof. The proof of (a) is a simple and standard perturbation argument (if \( p^*_n \) did not satisfy (a) then it would be possible to perturb \( p^*_n \) to reduce its norm on \( [-1,1] \) without decreasing the derivative at 1). We will prove only that \( p_n \) has no roots in \( (1,\infty) \), the other parts are similar. First suppose that \( p^*_n \) has two roots at \( \alpha > 1 \) and \( \beta = 1 \). Consider
\[
v_n(x) := \frac{p^*_n(x)(x - 1)^2}{(x - \alpha)(x - \beta)}.
\]
Then for sufficiently small \( \varepsilon > 0 \):
(i) \( \|p^*_n - \varepsilon v_n\|_{[-1,1]} < \|p^*_n\|_{[-1,1]} \),
(ii) \( (p^*_n - \varepsilon v_n)'(1) = |p^*_n'(1)| \),
(iii) \( p^*_n - \varepsilon v_n \) has all but at most \( k \) roots in \( \mathbb{R} - (1,1) \).

Part (iii) follows since \( (x - \alpha)(x - \beta) - \varepsilon(x - 1)^2 \) has two roots in \( [1,\infty) \) for sufficiently small \( \varepsilon \). (Note \( \alpha \) may equal \( \beta \).) However, this contradicts the maximality of \( p^*_n \).

Next we suppose that \( p^*_n \) has exactly one (nonrepeat) root at \( \alpha > 1 \). Now we argue as before by considering
\[
v_n(x) := \frac{p^*_n(x)(x - 1)^2}{(x - 1)(x - \alpha)}.
\]
If $p_n^*(1) \neq 0$ we must observe that in this case \( \text{sign}(p_n^*(1)) = \text{sign}(p_n^*(1)) \) and, hence, that

$$u_n'(1) = \frac{(1 - \alpha) p_n^*(1)}{(1 - \alpha)^2}$$

has the opposite sign to $p_n^*(1)$. The last observation requires noticing that if $p_n^*(1)$ has opposite sign to $p_n^*(1)$, then $p_n''$ has all its zeros in $(-\infty, 1]$ and, hence, $p_n''(1) \neq 0$. Thus, $|p_n^*|$ is increasing on $[1, \infty)$, $|p_n|$ is decreasing on $[1, \alpha)$ and $p_n^*(x + (\alpha - 1))$ violates the maximality assumptions on $[-1, 1]$.

Part (b) follows since if $p_n^*$ has two or more zeros at 1, then $p_n^*(1)$ would also equal zero.

\[\square\]

**Lemma 4.** If $p_n \in \pi_n$ has at least $(n - k)$ roots in $\mathbb{R} - (-1, 1)$, then

$$\left|p_n'(1)\right| \leq \frac{k}{2} (k + 1) n \|p_n\|_{[-1, 1]}.$$

**Proof.** If $2k \geq n$, then the lemma follows from Markov’s inequality, so we may suppose $2k < n$. Suppose there exists $p_n$, as above, so that

$$\left|p_n'(1)\right| > \frac{k}{2} (k + 1) (n - k) \|p_n\|_{[-1, 1]},$$

and let $q_n$ be the maximal such $p_n$. By Lemma 3, this $q_n$ equioscillates $k + 1$ times on $[-1, 1]$.

We shall first consider the case where $q_n$ has no root at 1. The key to the proof is to observe that the roots of $q_n$ lie to the left of the roots of $s_n$ (as defined in Lemma 3). We may write

$$q_n(x) = (x + 1)^{n-k} \prod_{i=1}^{k} (x - \rho_i),$$

where $-1 < \rho_1 < \cdots < \rho_k < 1$. Also,

$$s_n(x) = (x + 1)^{n-k} \prod_{i=1}^{k} (x - \gamma_i).$$

The claim is that $\gamma_i \geq \rho_i$ for each $i$. This is seen as follows. Choose the largest $i$ for which $\rho_i > \gamma_i$. Then pick $\eta$ so that $\|\eta q_n\|_{[\gamma_i, 1]} = \|s_n\|_{[\gamma_i, 1]}$. (We will specify the sign of $\eta$ later.) We can deduce from the equioscillation of $q_n$ that $\eta q_n - s_n$ has at least $k - i$ roots on $[\beta, 1)$, where $\beta$ is the first point greater than $\rho_i$ where $\eta q_n$ achieves its maximum modulus. From Lemma 2(c) we deduce that $\eta q_n - s_n$ has at least $i - 1$ roots on $(-1, \alpha)$, where $\alpha$ is the largest point less than $\gamma_i$ where $s_n$ achieves its maximum modulus. We need only observe that if we choose the sign of $\eta$ so that

$$\text{sign } \eta q_n(\beta) = -\text{sign } s_n(\alpha),$$

then $\eta q_n - s_n$ must have 2 roots in $(\alpha, \beta)$. Thus, $\eta q_n - s_n$ has $n + 1$ roots which is a contradiction and we conclude that $\rho_i \leq \gamma_i$. 

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We now observe that, since \( \rho_i \leq \gamma_i < 1 \),
\[
\frac{|q_n(1)|}{\|q_n\|_{[-1,1]}} = \frac{q_n'(1)}{q_n(1)} = \sum_{i=1}^{k} \frac{1}{1 - \rho_i} + \sum_{i=1}^{n-k} \frac{1}{1 - (-1)}
\leq \sum_{i=1}^{k} \frac{1}{1 - \gamma_i} + \frac{n-k}{2} \leq 4k(n-k) + \frac{n-k}{2},
\]
where the later inequality follows from Lemma 2. This is a contradiction.

In the case where \( q_n \) has exactly one root at 1 we proceed as follows. Let \( d > 1 \) be the unique point in \((1, \infty)\), where \( |q_n(d)| = \|q_n\|_{[-1,1]} \). We can now consider \( q_n \) on \([-1, d]\). We note that
\[
\frac{|q_n'(d)|}{\|q_n\|_{[-1,1]}} \geq \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}
\]
since \( |q_n'| \) is increasing on \([1, \infty)\). We can repeat verbatim the argument of the first part applied to
\[
\tilde{q}(x) = q_n \left( x \left( \frac{d+1}{2} \right) + \left( \frac{d-1}{2} \right) \right)
\]
with \( k \) replaced by \( k + 1 \). This allows us to deduce the contradiction that
\[
\frac{|q'(1)|}{\|q_n\|_{[-1,1]}} \leq \frac{|\tilde{q}'(1)|}{\|	ilde{q}_n\|_{[-1,1]}} \leq 4(k+1)(n-k) + \frac{n-k}{2}.
\]

The proof of the Conjecture is now straightforward.

**Proof of Conjecture.** Let \( p_n \) be a polynomial of degree \( n \) with \( n-k \) roots in \( \mathbb{R} \). Let \( x_0 \) be a point in \([-1,1]\), where \( p_n'(x_0) \) achieves its maximum modulus. We suppose \( x_0 < 0 \) (\( x_0 > 0 \) follows analogously). Let \( ax + b \) map \([x_0,1]\) one-to-one onto \([-1,1]\) in such a way that \( x_0 \to 1 \).

Note that \( |a| < 2 \). Thus, if \( v_n(ax+b) = p_n(x) \), then
\[
\frac{|p_n'(x_0)|}{\|p_n\|_{[-1,1]}} \leq \frac{2|v_n'(1)|}{\|v_n\|_{[-1,1]}} = 9n(k+1),
\]
where the last inequality follows from Lemma 4 and the observation that \( v_n \) has at least as many roots as \( p_n \) in \( \mathbb{R} \).

**References**


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