PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 1, January 1985

# COMMUTATIVE RANGES OF ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

R. J. FLEMING AND J. E. JAMISON

ABSTRACT. Let $A$ denote a complex unital Banach algebra with Hermitian elements $H(A)$. We show that if $F$ is an analytic function from a connected open set $D$ into $A$ such that $F(z)$ is normal ($F(z) = u(z) + iv(z)$, where $u(z)$, $v(z) \in H(A)$ and $u(z)v(z) = v(z)u(z)$) for each $z \in D$, then $F(z)F(w) = F(w)F(z)$ for all $w$, $z \in D$. This generalizes a theorem of Globevnik and Vidav concerning operator-valued analytic functions. As a corollary, it follows that an essentially normal-valued analytic function has an essentially commutative range.

In this note we extend to the Banach algebra setting a theorem of Globevnik and Vidav [7] which states that the set of values of a normal-operator-valued function, defined and analytic on an open connected set in the complex plane, is commutative.

Suppose $A$ is a complex unital Banach algebra with Hermitian elements $H(A)$. Recall that an element $u \in A$ is said to be *Hermitian* if its numerical range $V(u) = \{f(u): f(1) = 1, \|f\| = 1, f \in A^*\}$ lies on the real axis. An element $w \in A$ is said to be normal if $w = u + iv$, where $u$, $v \in H(A)$ and $uv = vu$. We let $J(A) = H(A) + iH(A)$. Then $J(A)$ is a Banach subspace of $A$ (but not necessarily a subalgebra) which contains both the Hermitian and normal elements. (See [4] for a discussion of these notions.) The mapping $*$ from $J(A)$ to itself, defined by $(u + iv)^* = u - iv$, is a continuous linear involution on $J(A)$. Furthermore, an element $w$ of $J(A)$ is normal if and only if $w^*w = ww^*$. Suppose that $F: D \to A$, where $D$ is the open unit disk, is analytic and normal valued for each $z \in D$. Thus $F(z) = \sum_{j=0}^{\infty} u_j z^j \in J(A)$ for each $z \in D$, where $u_j \in A$ for each $j$.

THEOREM. *Let $A$ be a unital Banach algebra. Let $F$ be analytic on the open unit disk $D$ and let $F(z) \in A$ be normal for all $z \in D$. Then there exist commuting normal elements $u_j \in A$ satisfying*

$$F(z) = \sum_{j=1}^{\infty} z^j u_j.$$

*In particular, $F(z)F(w) = F(w)F(z)$ holds for all $z$, $w \in D$.*

LEMMA. *If $F(z) = \sum_{j=0}^{\infty} u_j z^j$ is analytic on $D$ to $J(A)$, then $F^*(z) = \sum_{j=0}^{\infty} u_j^* \bar{z}^j$ (where $\bar{z}$ denotes the complex conjugate of $z$) converges absolutely for all $z \in D$.*

Received by the editors November 7, 1983 and, in revised form, February 21, 1984.
1980 *Mathematics Subject Classification*. Primary 46H10; Secondary 47D30.
*Key words and phrases*. Unital Banach algebras, Hermitian elements, normal elements, essentially normal.

PROOF. If $G$ is any differentiable function from $D$ to $J(A)$, then $G'(z) \in J(A)$ since $J(A)$ is a closed subspace of $A$. Hence, $F'(z), F''(z), \ldots, F^{(n)}(z), \cdots$ are in $J(A)$ for all $z \in D$. In particular, for each $m$, $u_m = F^{(m)}(0)/m! \in J(A)$. Since $\sum_{j=0}^{\infty} \|u_j\| r^j$ converges for all $r$, $0 \leqslant r < 1$, and since $\|u_j^*\| \leqslant 2e\|u_j\|$ [4, p. 50], it follows that $\sum_{j=0}^{\infty} \|u_j^*\| r^j$ converges for all $r$, $0 \leqslant r < 1$. This implies the absolute convergence of $\sum_{j=0}^{\infty} u_j^* \bar{z}^j$ for $z \in D$. Now $F_n(z) = \sum_{j=0}^{n} u_j z^j \in J(A)$ for each $z \in D$ and $F_n^*(z) = \sum_{j=0}^{n} u_j^* \bar{z}^j$. Thus

$$F^*(z) = \left(\lim_n F_n(z)\right)^* = \lim_n F_n^*(z) = \sum_{j=0}^{\infty} u_j^* \bar{z}^j.$$

The proof of the Theorem follows as in [7] except that the version of Fuglede's theorem due to Berkson, Dowson and Elliott [3] must be used.

REMARK. In the theorem it is enough to assume that $D$ is any open connected subset of the plane and $F(z)$ is analytic on $D$ and normal valued in a neighborhood of a point in $D$. By using the principle of analytic continuation exactly as in [7], it can be shown that the normality and commutativity must hold throughout $D$.

APPLICATIONS. (i) The Globevnik-Vidav theorem is a special case of the Theorem where $A = B(H)$, the bounded operators on a Hilbert space $H$.

(ii) The Theorem holds for $A = B(X)$, where $X$ is a Banach space. In this case, of course, the collection of normal operators may be rather limited. Characterizations of Hermitian operators (and hence of normal operators) for certain special Banach spaces may be found, for example, in [5 or 6].

(iii) If $F(z)$ is essentially normal valued, then the range of $F(z)$ is essentially commutative. In this case, we assume that $F(z)$ is analytic valued in $B(H)$ and essentially normal for each $z$. Hence the canonical image $\phi(F(z))$ is normal in the Calkin algebra $A = B(H)/K$, where $K$ denotes the ideal of compact operators. We conclude from the Theorem that the elements $\phi(F(z))$ commute in $A$ so that $F(z)F(w) - F(w)F(z)$ is compact for all $z, w$ in $D$.

(iv) More generally, if for all $z \in D$, $\phi(F(z))$ is normal in $A = B(X)/K$, where $X$ is a Banach space and $K$ is a closed two-sided ideal in $B(X)$, then $F(z)F(w) - F(w)F(z) \in K$ for all $w, z \in D$.

(v) The form of an operator $F(z)$ for which $\phi(F(z))$ is normal in $B(X)/K$ is not always clear. It is true that $\phi(F(z))$ is normal if

$$F(z) = w(z) + k(z), \tag{1}$$

where $w(z)$ is normal in $B(X)$ and $k(z) \in K$, or, even more generally, when

$$F(z) = u(z) + iv(z), \tag{2}$$

where $u(z)v(z) - v(z)u(z) \in K$, and each of $u(z)$ and $v(z)$ is the sum of an Hermitian element of $B(X)$ and an element of $K$.

It is not always the case that $\phi(F(z))$ normal implies that $F(z)$ is of the form (1) or (2). This question is one of whether the coset "lifts" to an operator of the form (1) or (2). In the case where $X = L_p$ or $l_p$, $1 < p < \infty$, and $K$ is the set of compact operators, the answer is known to be yes [1, 2].

It would perhaps be of interest to know whether commutativity of the range of $F(z)$ must continue to hold if the assumption of normality in the Theorem is relaxed to some weaker form such as quasinormality or subnormality.

## REFERENCES

1. G. D. Allen and J. D. Ward, *Hermitian liftings in $B(l_p)$*, J. Operator Theory **1** (1979), 27–36.

2. G. D. Allen, D. A. Legg and J. D. Ward, *Essentially Hermitian operators in $B(L_p)$*, Proc. Amer. Math. Soc. **80** (1980), 71–77.

3. E. Berkson, H. R. Dowson and G. A. Elliott, *On Fuglede's theorem and scalar type operators*, Bull. London Math. Soc. **4** (1972), 13–16.

4. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras*, Cambridge Univ. Press, London, 1971.

5. _____, *Numerical ranges*. II, Cambridge Univ. Press, London, 1973.

6. R. J. Fleming and J. E. Jamison, *Hermitian and adjoint abelian operators on certain Banach spaces*, Pacific J. Math. **52** (1974), 67–84.

7. J. Globevnik and I. Vidav, *A note on normal-operator-valued analytic functions*, Proc. Amer. Math. Soc. **37** (1973), 619–621.

DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT, MICHIGAN 48859

DEPARTMENT OF MATHEMATICS SCIENCES, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152