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# ON THE ELLIPTIC EQUATION $\Delta u = \phi(x)u^\gamma$ IN $R^2$

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ABSTRACT. The equation $(*)$ $\Delta u = \phi(x)u^\gamma$ is considered in $R^2$, where $\gamma \neq 1$ and $\phi(x) \geq 0$ is locally Hölder continuous. Sufficient conditions are obtained for $(*)$ to possess infinitely many positive solutions which are defined throughout $R^2$ and have logarithmic growth as $|x| \to \infty$. An extension of the main result to the higher-dimensional case is also attempted.

**1. Introduction.** Consider the two-dimensional elliptic equation

$$\Delta u = \phi(x)u^\gamma, \tag{1}$$

where $x = (x_1, x_2)$, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, under the conditions:

(a) $\gamma$ is a positive constant with $\gamma \neq 1$;

(b) $\phi$: $R^2 \to R_+$, $R_+ = [0, \infty)$, is a locally Hölder continuous (with exponent $\theta \in (0,1)$) function which is positive in some neighborhood of the origin.

We are interested in the existence of positive entire solutions of (1). By an entire solution of (1) we mean a function $u \in C^{2+\theta}_{\mathrm{loc}}(R^2)$ which satisfies (1) at every point of $R^2$. From Liouville's theorem on subharmonic functions in $R^2$ it follows that a positive entire solution of (1), if any, is necessarily unbounded. Our objective here is to establish conditions for equation (1) to have positive entire solutions with logarithmic growth at infinity. We employ the supersolution-subsolution method and show that suitable supersolutions and subsolutions depending only on $|x| = (x_1^2 + x_2^2)^{1/2}$ can be constructed with the help of an existence theory of second order nonlinear ordinary differential equations. An extension of the main result to the higher-dimensional case is also given; it is shown that there exists a class of equations of the form (1) in $R^n$, $n \geq 3$, having unbounded positive entire solutions.

For closely related results on the existence of entire solutions we refer the reader to the papers **[2, 3, 4]** in which equation (1) in higher dimensions and the equation $\Delta u = \phi(x)e^u$ are studied.

**2. Main result.** We use the notation

$$\phi^*(t) = \max_{|x|=t} \phi(x), \qquad \phi_*(t) = \min_{|x|=t} \phi(x). \tag{2}$$

The main result of this paper is the following

THEOREM 1. *In addition to conditions* (a) *and* (b) *suppose that*

$$\int_1^\infty t(\log t)^\gamma \phi^*(t)\, dt < \infty. \tag{3}$$

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*Then equation* (1) *has infinitely many positive entire solutions in* $R^2$ *with logarithmic growth at infinity.*

The proof of this theorem is based on the following lemma which is a special case of Theorem 2.10 of Ni [**3**].

LEMMA. *If there exist positive functions* $v, w \in C^{2+\theta}_{\mathrm{loc}}(R^2)$ *such that*

$$\Delta v \geq \phi(x) v^\gamma, \qquad x \in R^2, \tag{4}$$

$$\Delta w \leq \phi(x) w^\gamma, \qquad x \in R^2, \tag{5}$$

$$v(x) \leq w(x), \qquad x \in R^2, \tag{6}$$

*then equation* (1) *has an entire solution* $u(x)$ *satisfying*

$$v(x) \leq u(x) \leq w(x), \qquad x \in R^2. \tag{7}$$

A function $v$ [resp. $w$] satisfying (4) [resp. (5)] is called a subsolution [resp. supersolution] of (1) in $R^2$.

PROOF OF THEOREM 1. Since $\phi_*(|x|) \leq \phi(x) \leq \phi^*(|x|)$, $x \in R^2$, a positive solution $v(x)$ of the equation

$$\Delta v = \phi^*(|x|) v^\gamma, \qquad x \in R^2, \tag{8}$$

is a subsolution of (1) in $R^2$, and a positive solution $w(x)$ of the equation

$$\Delta w = \phi_*(|x|) w^\gamma, \qquad x \in R^2, \tag{9}$$

is a supersolution of (1) in $R^2$. It is natural to seek solutions of (8) and (9) depending only on $|x|$. If we put $v(x) = y(|x|)$ and $w(x) = z(|x|)$, then we are led to the following one-dimensional initial value problems for $y(t)$ and $z(t)$:

$$y'' + \frac{1}{t} y' = \phi^*(t) y^\gamma, \quad t > 0, \qquad y(0) = \alpha, \qquad y'(0) = 0, \tag{10}$$

$$z'' + \frac{1}{t} z' = \phi_*(t) z^\gamma, \quad t > 0, \qquad z(0) = \beta, \qquad z'(0) = 0, \tag{11}$$

where $' = d/dt$, and $\alpha$ and $\beta$ are positive constants.

We first solve (10) by transforming it into the integral equation

$$y(t) = \alpha + \int_0^t s \log(t/s) \cdot \phi^*(s) y^\gamma(s)\, ds, \qquad t \geq 0. \tag{12}$$

Choose a constant $\alpha > 0$ (small enough if $\gamma > 1$ and large enough if $\gamma < 1$) so that

$$(2\alpha)^\gamma \int_0^e \phi^*(s)\, ds \leq \alpha/3, \tag{13}$$

$$2^{\gamma-1} \alpha^\gamma (e^2 - 1) \max_{1 \leq s \leq e} \phi^*(s) \leq \alpha/3, \tag{14}$$

$$(2\alpha)^\gamma \int_e^\infty s(\log s)^\gamma \phi^*(s)\, ds \leq \alpha/3, \tag{15}$$

and define the function $A_\alpha(t)$ by

$$A_\alpha(t) = 2\alpha \quad \text{for } 0 \leq t \leq e, \qquad A_\alpha(t) = 2\alpha \log t \quad \text{for } t \geq e. \tag{16}$$

Let $\mathcal{C}$ denote the locally convex space of all continuous functions on $[0,\infty)$ with the usual topology and consider the set

$$Y = \{y \in \mathcal{C}\colon\ \alpha \le y(t) \le A_\alpha(t) \text{ for } t \ge 0\}. \tag{17}$$

Clearly, $Y$ is a closed convex subset of $\mathcal{C}$. Define the mapping $\mathcal{F}$ by

$$\mathcal{F}y(t) = \alpha + \int_0^t s\,\log(t/s)\cdot\phi^*(s)y^\gamma(s)\,ds, \qquad t \ge 0. \tag{18}$$

Then it can be verified that $\mathcal{F}$ is a continuous mapping from $Y$ into itself such that $\mathcal{F}Y$ is relatively compact.

(i) *$\mathcal{F}$ maps $Y$ into itself.* It suffices to show that $y \in Y$ implies $\mathcal{F}y(t) \le A_\alpha(t)$ for $t \ge 0$. Let $0 \le t \le e$. Then, using (13) and the inequality $0 \le s\,\log(t/s) \le t/e$ for $0 \le s \le t$, we have

$$\mathcal{F}y(t) \le \alpha + (t/e)\int_0^t \phi^*(s)y^\gamma(s)\,ds \le \alpha + (2\alpha)^\gamma\int_0^e \phi^*(s)\,ds \le \alpha + (\alpha/3) < 2\alpha.$$

Let $t \ge e$. We then write

$$\begin{aligned}\mathcal{F}y(t) &= \alpha + \left(\int_0^1 + \int_1^e + \int_e^t\right) s\,\log(t/s)\cdot\phi^*(s)y^\gamma(s)\,ds\\ &= \alpha + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned} \tag{19}$$

The inequality $0 \le s\,\log(t/s) \le \log t$ for $0 \le s \le 1$ together with (13) implies that

$$\mathrm{I}_1 \le \int_0^1 \phi^*(s)y^\gamma(s)\,ds\cdot\log t \le (2\alpha)^\gamma\int_0^1 \phi^*(s)\,ds\cdot\log t \le (\alpha/3)\,\log t. \tag{20}$$

With the use of (14) and (15) the integrals $\mathrm{I}_2$ and $\mathrm{I}_3$ are estimated as follows:

$$\begin{aligned}\mathrm{I}_2 &\le (2\alpha)^\gamma\int_1^e s\,\log(t/s)\cdot\phi^*(s)\,ds \le (2\alpha)^\gamma \max_{1\le s\le e}\phi^*(s)\int_1^e s\,ds\cdot\log t\\ &= 2^{\gamma-1}\alpha^\gamma(e^2-1)\max_{1\le s\le e}\phi^*(s)\cdot\log t \le (\alpha/3)\,\log t,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathrm{I}_3 &\le (2\alpha)^\gamma\int_e^t s\,\log(t/s)\cdot\phi^*(s)(\log s)^\gamma\,ds\\ &\le (2\alpha)^\gamma\int_e^t s(\log s)^\gamma\phi^*(s)\,ds\cdot\log t\\ &\le (2\alpha)^\gamma\int_e^\infty s(\log s)^\gamma\phi^*(s)\,ds\cdot\log t \le (\alpha/3)\,\log t.\end{aligned} \tag{22}$$

From (20)–(22) it follows that $\mathcal{F}y(t) \le 2\alpha\,\log t$ for $t \ge e$. We have thus shown that $\mathcal{F}y(t) \le A_\alpha(t)$ for $t \ge 0$ as desired.

(ii) *$\mathcal{F}$ is continuous.* Let $\{y_m\}$ be a sequence in $Y$ converging to $y \in Y$ as $m \to \infty$ in the topology of $\mathcal{C}$. Then we have

$$|\mathcal{F}y_m(t) - \mathcal{F}y(t)| \le (t/e)\int_0^t \phi^*(s)|y_m^\gamma(s) - y^\gamma(s)|\,ds, \qquad t \ge 0.$$

Since the functions $f_m(s) = \phi^*(s)|y_m^\gamma(s) - y^\gamma(s)|$ satisfy $f_m(s) \le 2\phi^*(s)A_\alpha^\gamma(s)$, $s \ge 0$, and $f_m(s) \to 0$ as $m \to \infty$ at every point $s \ge 0$, the Lebesgue dominated convergence theorem implies that $\mathcal{F}y_m(t) \to \mathcal{F}y(t)$ as $m \to \infty$ uniformly on compact subintervals of $[0,\infty)$. This shows that $\mathcal{F}y_m \to \mathcal{F}y$ in $\mathcal{C}$ as $m \to \infty$.

(iii) $\mathcal{F}Y$ *is relatively compact.* It suffices to prove that $\mathcal{F}Y$ is uniformly bounded and equicontinuous at every point of $[0,\infty)$. But this is an immediate consequence of the following inequalities holding for all $y \in Y$ and all $t \geq 0$: $\alpha \leq \mathcal{F}y(t) \leq A_\alpha(t)$, and

$$(\mathcal{F}y)'(t) = (1/t)\int_0^t s\phi^*(s)y^\gamma(s)\,ds \leq \int_0^t \phi^*(s)A_\alpha^\gamma(s)\,ds.$$

Thus we are able to apply the Schauder-Tychonoff fixed point theorem (cf. e.g. [**1**, p. 161]), concluding that $\mathcal{F}$ has a fixed point $y \in Y$. This fixed point $y = y(t)$ is a solution of (12), and hence a solution of (10) on $[0,\infty)$. The function $v(x) = y(|x|)$ gives a subsolution of (1) in $R^2$.

Next, choose a constant $\beta > 0$ so that the inequalities (13)–(15) with $\alpha$ and $\phi^*(s)$ replaced by $\beta$ and $\phi_*(s)$, respectively, are satisfied and consider the mapping $\mathcal{G}$ defined by

$$\mathcal{G}z(t) = \beta + \int_0^t s\,\log(t/s)\cdot\phi_*(s)z^\gamma(s)\,ds, \qquad t \geq 0. \tag{23}$$

Arguing exactly as above, we can show that $\mathcal{G}$ has a fixed point $z$ in the set

$$Z = \{z \in \mathcal{C}\colon\ \beta \leq z(t) \leq A_\beta(t) \text{ for } t \geq 0\}, \tag{24}$$

where $A_\beta(t)$ is defined by (16) with $\alpha$ replaced by $\beta$. This fixed point $z = z(t)$ is a solution of (11), giving rise to a supersolution $w(x) = z(|x|)$ of (1) in $R^2$.

Finally, it is easy to see that $\alpha > 0$ and $\beta > 0$ can be chosen so that, in addition to the conditions required above, the inequalities

$$2\alpha < \beta^\gamma \int_0^e s\phi_*(s)\,ds < \beta \tag{25}$$

are satisfied. With this choice of $\alpha$ and $\beta$ the functions $y(t)$ and $z(t)$ satisfy $y(t) \leq z(t)$ for $t \geq 0$. In fact, if $0 \leq t \leq e$, then $\alpha \leq y(t) \leq 2\alpha < \beta \leq z(t) < 2\beta$. If $t \geq e$, then rewriting the differential equation in (11) as $(tz')' = t\phi_*(t)z^\gamma$, we see that $(tz'(t))' \geq 0$ for $t > 0$, whence we have $tz'(t) \geq ez'(e)$ for $t \geq e$. We now integrate this inequality from $e$ to $t$. Noting that

$$z'(e) = (1/e)\int_0^e s\phi_*(s)z^\gamma(s)\,ds$$

and using (25), we obtain

$$\begin{aligned} z(t) &\geq z(e) + ez'(e)\int_e^t \frac{ds}{s} \\ &= z(e) + \int_0^e s\phi_*(s)z^\gamma(s)\,ds\cdot(\log t - 1) \\ &\geq \beta + \beta^\gamma\int_0^e s\phi_*(s)\,ds\cdot(\log t - 1) \\ &> \beta^\gamma\int_0^e s\phi_*(s)\,ds\cdot\log t \\ &> 2\alpha\,\log t \geq y(t), \qquad t \geq e. \end{aligned} \tag{26}$$

It follows therefore that the subsolution $v(x) = y(|x|)$ and the supersolution $w(x) = z(|x|)$ satisfy $v(x) \leq w(x)$ in $R^2$, and so the lemma guarantees the existence of an entire solution $u(x)$ of (1) lying between $v(x)$ and $w(x)$ in $R^2$. In view of the inequality

$$y(t) \geq \alpha + \alpha^\gamma \int_0^e s\phi^*(s)\,ds \cdot (\log t - 1), \qquad t \geq e, \tag{27}$$

which can be derived exactly as above, we conclude that the solution $u(x)$ obtained has logarithmic growth as $|x| \to \infty$, that is, there exist positive constants $k_1, k_2$ and $R > 1$ such that

$$k_1 \log|x| \leq u(x) \leq k_2 \log|x| \quad \text{for } |x| \geq R. \tag{28}$$

To complete the proof it suffices to observe that a sequence of pairs of positive numbers, $\{(\alpha_k, \beta_k)\}_{k=1}^\infty$, can be chosen so that the corresponding entire solutions $\{u_k(x)\}_{k=1}^\infty$ are distinct.

REMARK. The condition (3) in Theorem 1 is sharp in the sense that in case $\phi(x)$ is such that

$$\sup_{t\geq 0} \frac{\phi^*(t)}{\phi_*(t)} < \infty, \tag{29}$$

it is both necessary and sufficient in order that equation (1) possess positive entire solutions with logarithmic growth as $|x| \to \infty$. To see this, let $u(x)$ be a positive solution of (1) in $R^2$ satisfying (28) for some positive constants $k_1$, $k_2$ and $R > 1$. Taking the average of (1) on a circle $|x| = t$ and using (28), we see that the function $U(t) = \int_{|x|=t} u(x)\,d\sigma/2\pi t$ satisfies

$$t^{-1}(tU'(t))' \geq \phi_*(t) \int_{|x|=t} u^\gamma(x)\,d\sigma/2\pi t \geq k_1^\gamma (\log t)^\gamma \phi_*(t), \qquad t \geq R. \tag{30}$$

Since $U'(t) > 0$ and $tU'(t)$ is bounded above, integrating (30) yields

$$\int_R^\infty t(\log t)^\gamma \phi_*(t)\,dt < \infty, \tag{31}$$

which is equivalent to (3) if (29) holds.

**3. Extension to higher dimensions.** Using a device due to Ni [**3, 4**], we can derive from Theorem 1 an existence result for the elliptic equation

$$\Delta u = \phi(x)u^\gamma \tag{32}$$

in $R^n$ with $n \geq 3$, where $x = (x_1, \ldots, x_n)$, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$, $\gamma\ (\neq 1)$ is a positive constant, and $\phi$: $R^n \to R_+$ is a locally Hölder continuous (with exponent $\theta \in (0,1)$) function in $R^n$.

In what follows we write $x = (x', x'') \in R^2 \times R^{n-2} = R^n$.

THEOREM 2. *Suppose that there exist locally Hölder continuous functions $\phi^*$, $\phi_*: R_+ \to R_+$ such that $\phi_*(t) > 0$ in some interval $[0, \tau]$, $\tau > 0$, and*

$$\phi_*(|x'|) \leq \phi(x) \leq \phi^*(|x'|) \quad \textit{for all } x \in R^n. \tag{33}$$

*If condition* (3) *holds with this $\phi^*$, then equation* (32) *has infinitely many entire solutions which are positive and unbounded in $R^n$.*

PROOF. Consider the equations

$$\Delta' \tilde{v} = \phi^*(|x'|)\tilde{v}^\gamma, \tag{34}$$

$$\Delta' \tilde{w} = \phi_*(|x'|)\tilde{w}^\gamma \tag{35}$$

in $R^2$, where $\Delta'$ denotes the two-dimensional Laplacian taken with respect to $x'$. From the proof of Theorem 1 we see that (34) and (35) have unbounded positive entire solutions $\tilde{v}(x')$ and $\tilde{w}(x')$ depending only on $|x'|$ and such that $\tilde{v}(x') \leq \tilde{w}(x')$ for $x' \in R^2$. Define

$$v(x) = \tilde{v}(x') \quad \text{and} \quad w(x) = \tilde{w}(x'). \tag{36}$$

Then, in view of (33), we obtain

$$\Delta v - \phi(x)v^\gamma \geq \Delta'\tilde{v} - \phi^*(|x'|)\tilde{v}^\gamma = 0,$$
$$\Delta w - \phi(x)w^\gamma \leq \Delta'\tilde{w} - \phi_*(|x'|)\tilde{w}^\gamma = 0$$

in $R^n$, which shows that $v(x)$ and $w(x)$ are, respectively, a subsolution and a supersolution of (32) in $R^n$. Since $v(x) \leq w(x)$ in $R^n$, from the $n$-dimensional version of the existence lemma it follows that (32) has an entire solution $u(x)$ such that $\tilde{v}(x') \leq u(x) \leq \tilde{w}(x')$ in $R^n$. It is obvious that

$$k_1 \log|x'| \leq u(x) \leq k_2 \log|x'|, \qquad |x'| \geq R, \text{ uniformly in } x'',$$

for some positive constants $k_1, k_2$ and $R > 1$. This finishes the proof.

EXAMPLE. Consider the equations

$$\Delta u = c(1 + |x'|)^\lambda u^\gamma, \tag{37}$$

$$\Delta u = c(1 + |x|)^\lambda u^\gamma \tag{38}$$

in $R^n$, $n \geq 3$, where $x'$ and $\gamma$ are as before, and $c > 0$ and $\lambda$ are constants. If $\lambda < -2$, then by Theorem 2 equation (37) has infinitely many unbounded positive entire solutions, while equation (38) admits infinitely many positive entire solutions which are bounded and tend to positive constants as $|x| \to \infty$ (see [**2, 3**]).

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