ITERATES OF HOLOMORPHIC SELF-MAPS
OF THE UNIT BALL IN HILBERT SPACE

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ABSTRACT. An example is given of a biholomorphic self-mapping \( T \) of the unit ball in infinite-dimensional Hilbert space satisfying

\[
0 = \lim \inf_n \|T^n(0)\| < \lim \sup_n \|T^n(0)\| = 1.
\]

1. Introduction. Let \( B \) denote an open unit ball in the complex Hilbert space \( H \) and let \( T \) be a holomorphic self-mapping of the ball \( B \) with no fixed points in \( B \). The sequence of iterates of such a mapping was considered by several authors. In case \( H = \mathbb{C} \) (the complex plane) the basic result comes from the old papers of J. Wolff [15] and A. Denjoy [3] (see also [1]). They showed that the iterates converge to a unimodular constant (uniformly on compact subsets of the unit disc \( \Delta \)). Recently it was generalized [11, 13] to self-mappings of the unit ball \( B \subset \mathbb{C}^n \) with \( n > 1 \).

In the infinite-dimensional case analogous results have been obtained in several special cases only [7, 14]. It was not quite clear whether or not a similar statement holds for all holomorphic, fixed point free mappings \( T: B \to B \). Our aim in this paper is to construct an example showing that the answer is negative even for automorphisms (i.e. biholomorphic self-mappings of the ball \( B \)).

Recall that \( B \) can be furnished with the invariant hyperbolic metric \( \rho \) so that every holomorphic mapping \( T: B \to B \) becomes \( \rho \)-nonexpansive [5, 10]. Recently it has been shown that there exist interesting analogies between properties of \( \rho \)-nonexpansive self-mappings of the ball \( B \) and of norm-nonexpansive mappings on the whole space \( H \) [6, 9, 12]. Therefore our question is closely related to a similar problem concerning a norm-nonexpansive, fixed point free mapping \( F: H \to H \). For such a mapping one can show that if \( \dim H < \infty \), then \( \lim_{n \to \infty} \|F^n(x)\| = \infty \) for all \( x \in H \) (see [2 and 16] for even more general results). On the other hand M. Edelstein [4] gave an example of an isometry \( F: l^2 \to l^2 \) satisfying the following conditions:

(i) \( F \) has no fixed point in \( l^2 \);
(ii) \( \{F^n(0)\} \) is norm-unbounded;
(iii) There exists a subsequence \( \{F^{n_k}(0)\} \) with \( \lim_{k \to \infty} F^{n_k}(0) = 0 \) (see [4] for details). Here \( l^2 \) denotes the space of all sequences \( x = (x_n) \) of complex numbers with the usual Hilbert norm. Let us notice that \( F \) can be easily rewritten in the form

\[
F(x) = f(x) + a,
\]
where \( x \in H, \ a = F(0) \) and \( f \) is a linear isometry onto \( l^2 \).

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Our construction below is based upon this example.

2. Construction of the example. Let the infinite-dimensional complex Hilbert space \( H \) be given. Recall first that the unit ball \( B \subset H \) is the biholomorphic image of the region (the Siegel upper half-space) \( \Omega = \{ (\lambda, w) \in \mathbb{C} \times H : \text{Im} \lambda > ||w||^2 \} \) under the Cayley transform \( C \) (see, for instance, [8, Chapter 2, §31]). Notice that \( C \) maps the point \((i,0) \in \Omega \) into the origin. Clearly every holomorphic mapping \( T: B \to B \) can be obtained from a suitable holomorphic mapping \( \Phi: \Omega \to \Omega \) by composition on the right and left by \( C^{-1} \) and \( C \) respectively.

Now let \( K \) be a real Hilbert space such that \( H = K \oplus iK \). Denote by \( F_1 \) an isometry of the space \( K \) which satisfies properties (i)-(iv) with \( l^2 \) replaced by \( K \) (the existence of such an isometry follows immediately from Edelstein’s example). Clearly \( F_1 \) can be extended to an isometry \( F: H \to H \) also satisfying (i)-(iv) and, in addition, satisfying

\[ F(K) = K. \]

Now we define the mapping \( \Phi: \Omega \to \Omega \) by

\[
\Phi(\lambda, w) = [\lambda + i||a||^2 + 2i<f(w), a>, f(w)]
\]

(\( \langle , \rangle \) denotes the inner product in \( H \)). It is easy to observe that \( \Phi \) is an automorphism of \( \Omega \). A little calculation shows that (1) can be rewritten in the form

\[
\Phi(\lambda, w) = [\lambda + i(\|F(w)\|^2 - ||w||^2) - 2\text{Im}(f(w), a), f(w)],
\]

which reduces for \( w \in K \) to

\[
\Phi(\lambda, w) = [\lambda + i(\|F(w)\|^2 - ||w||^2), F(w)].
\]

Hence we obtain

\[
\Phi^n(\lambda, w) = [\lambda + i(\|F^n(w)\|^2 - ||w||^2), F^n(w)]
\]

for every \( w \in K \), \( \text{Im} \lambda > ||w||^2 \) and \( n = 1, 2, \ldots \).

Set \((\lambda, w) = (i, 0) \) in (2). We have

\[
\Phi^n(i, 0) = [i + i\|F^n(0)\|^2, F^n(0)].
\]

From (ii) and (iii) it immediately follows that \( \Phi^n(i, 0) \to (i, 0) \) as \( k \to \infty \) and that \( \{\Phi^n(i, 0)\} \) is unbounded in \( \Omega \).

Put now \( T = C \circ \Phi \circ C^{-1} \). It is easy to observe that \( T \) is a fixed point free automorphism of the ball \( B \) such that \( \limsup_n \|T^n(0)\| = 1 \) and \( T^n(0) \to 0 \) as \( k \to \infty \). Thus \( T \) is the desired example.

REFERENCES


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