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ITERATES OF HOLOMORPHIC SELF-MAPS OF THE UNIT BALL IN HILBERT SPACE

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ABSTRACT. An example is given of a biholomorphic self-mapping T of the unit ball in infinite-dimensional Hilbert space satisfying $0 = \liminf_n \|T^n(0)\| < \limsup_n \|T^n(0)\| = 1$.

1. Introduction. Let B denote an open unit ball in the complex Hilbert space H and let T be a holomorphic self-mapping of the ball B with no fixed points in B. The sequence of iterates of such a mapping was considered by several authors. In case $H = \mathbf{C}$ (the complex plane) the basic result comes from the old papers of J. Wolff [**15**] and A. Denjoy [**3**] (see also [**1**]). They showed that the iterates converge to a unimodular constant (uniformly on compact subsets of the unit disc Δ). Recently it was generalized [**11, 13**] to self-mappings of the unit ball $B \subset \mathbf{C}^n$ with $n > 1$.

In the infinite-dimensional case analogous results have been obtained in several special cases only [**7, 14**]. It was not quite clear whether or not a similar statement holds for all holomorphic, fixed point free mappings $T\colon B \to B$. Our aim in this paper is to construct an example showing that the answer is negative even for automorphisms (i.e. biholomorphic self-mappings of the ball B).

Recall that B can be furnished with the invariant hyperbolic metric ρ so that every holomorphic mapping $T\colon B \to B$ becomes ρ-nonexpansive [**5, 10**]. Recently it has been shown that there exist interesting analogies between properties of ρ-nonexpansive self-mappings of the ball B and of norm-nonexpansive mappings on the whole space H [**6, 9, 12**]. Therefore our question is closely related to a similar problem concerning a norm-nonexpansive, fixed point free mapping $F\colon H \to H$. For such a mapping one can show that if $\dim H < \infty$, then $\lim_{n\to\infty} \|F^n(x)\| = \infty$ for all x in H (see [**2** and **16**] for even more general results). On the other hand M. Edelstein [**4**] gave an example of an isometry $F\colon l^2 \to l^2$ satisfying the following conditions:

(i) F has no fixed point in l^2;

(ii) $\{F^n(0)\}$ is norm-unbounded;

(iii) There exists a subsequence $\{F^{n_k}(0)\}$ with $\lim_{k\to\infty} F^{n_k}(0) = 0$

(see [**4**] for details). Here l^2 denotes the space of all sequences $x = (x_n)$ of complex numbers with the usual Hilbert norm. Let us notice that F can be easily rewritten in the form

(iv) $F(x) = f(x) + a$,

where $x \in H$, $a = F(0)$ and f is a linear isometry onto l^2.

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Our construction below is based upon this example.

2. Construction of the example. Let the infinite-dimensional complex Hilbert space H be given. Recall first that the unit ball $B \subset H$ is the biholomorphic image of the region (the Siegel upper half-space) $\Omega = \{(\lambda, w) \in \mathbf{C} \times H\colon \operatorname{Im}\lambda > \|w\|^2\}$ under the Cayley transform C (see, for instance, [**8**, Chapter 2, §31]). Notice that C maps the point $(i, 0) \in \Omega$ into the origin. Clearly every holomorphic mapping $T\colon B \to B$ can be obtained from a suitable holomorphic mapping $\Phi\colon \Omega \to \Omega$ by composition on the right and left by C^{-1} and C respectively.

Now let K be a real Hilbert space such that $H = K \oplus iK$. Denote by F_1 an isometry of the space K which satisfies properties (i)–(iv) with l^2 replaced by K (the existence of such an isometry follows immediately from Edelstein's example). Clearly F_1 can be extended to an isometry $F\colon H \to H$ also satisfying (i)–(iv) and, in addition, satisfying

(v) $F(K) = K$.

Now we define the mapping $\Phi\colon \Omega \to \Omega$ by

$$\Phi(\lambda, w) = [\lambda + i\|a\|^2 + 2i\langle f(w), a\rangle, f(w)] \tag{1}$$

($\langle\ ,\ \rangle$ denotes the inner product in H). It is easy to observe that Φ is an automorphism of Ω. A little calculation shows that (1) can be rewritten in the form

$$\Phi(\lambda, w) = [\lambda + i(\|F(w)\|^2 - \|w\|^2) - 2\operatorname{Im}\langle f(w), a\rangle, F(w)],$$

which reduces for $w \in K$ to

$$\Phi(\lambda, w) = [\lambda + i(\|F(w)\|^2 - \|w\|^2), F(w)].$$

Hence we obtain

$$\Phi^n(\lambda, w) = [\lambda + i(\|F^n(w)\|^2 - \|w\|^2), F^n(w)] \tag{2}$$

for every $w \in K$, $\operatorname{Im}\lambda > \|w\|^2$ and $n = 1, 2, \ldots$.

Set $(\lambda, w) = (i, 0)$ in (2). We have

$$\Phi^n(i, 0) = [i + i\|F^n(0)\|^2, F^n(0)].$$

From (ii) and (iii) it immediately follows that $\Phi^{n_k}(i, 0) \to (i, 0)$ as $k \to \infty$ and that $\{\Phi^n(i, 0)\}$ is unbounded in Ω.

Put now $T = C \circ \Phi \circ C^{-1}$. It is easy to observe that T is a fixed point free automorphism of the ball B such that $\limsup_n \|T^n(0)\| = 1$ and $T^{n_k}(0) \to 0$ as $k \to \infty$. Thus T is the desired example.

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