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CONTRACTIONS WITH THE BICOMMUTANT PROPERTY

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ABSTRACT. It is shown that if T is a contraction for which there is an operator W with dense range such that $WT = SW$ for some unilateral shift S, then T has the bicommutant property, that is, the double commutant of T is the weakly closed algebra generated by T and the identity. As an example of such a contraction we have a contraction T such that $I - T^*T$ is of trace class and the spectrum of T fills the unit disc.

1. A bounded linear operator T on a Hilbert space is said to have the *bicommutant property* if $\{T\}'' = \operatorname{Alg} T$, where $\{T\}''$ and $\operatorname{Alg} T$ denote the double commutant of T and the weakly closed algebra generated by T and the identity, respectively. Every nonunitary isometry has the bicommutant property [**6**]. This result was extended by Uchiyama [**7** and **8**] and Wu [**12**] to some classes of contractions T whose *defect operators* $D_T = (I-T^*T)^{1/2}$ are of finite rank. In [**7** and **8**], Uchiyama proved the bicommutant property for $C_{\cdot 0}$-contractions not of class C_{00} whose defect operators are of finite rank. Subsequently Wu [**12**] proved this property for $C_{1\cdot}$-contractions not of class C_{11} whose defect operators are of finite rank. Recall that a contraction T is of class $C_{0\cdot}$ if $\lim_{n\to\infty} \|T^n x\| = 0$ for every x, of class $C_{1\cdot}$ if $\lim_{n\to\infty} \|T^n x\| \neq 0$ for every nonzero x, and that T is of class $C_{\cdot 0}$ (resp. $C_{\cdot 1}$) if T^* is of class $C_{0\cdot}$ (resp. $C_{1\cdot}$). For $\alpha, \beta = 0, 1$, $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. In this note the above results are extended to contractions whose defect operators are of Hilbert-Schmidt class; indeed, we obtain a more general result.

A contraction is *completely nonunitary* (c.n.u.) if it has no nontrivial unitary direct summand. Given a c.n.u. contraction T, the H^∞-functional calculus of Sz.-Nagy and Foias defines the operator $\phi(T)$ in $\operatorname{Alg} T$ for every ϕ in H^∞ (cf. [**2**, Chapter III]). In this note all Hilbert spaces are assumed to be separable.

Our main result is the following theorem, which we prove in §3.

THEOREM 1. *If T is a c.n.u. contraction and there exists an operator W with dense range such that $WT = SW$ for some unilateral shift S, then*

$$\{T\}'' = \{\phi(T) \colon \phi \in H^\infty\},$$

and in particular T has the bicommutant property.

A contraction T is called a *weak contraction* if its defect operator D_T is of Hilbert-Schmidt class and its spectrum $\sigma(T)$ does not fill the open unit disc D. A $C_{1\cdot}$-contraction with Hilbert-Schmidt defect operator is a weak contraction if and only if it is of class C_{11}, and a $C_{\cdot 0}$-contraction with Hilbert-Schmidt defect operator is a weak contraction if and only if it is of class C_{00} (cf. [**2**, Chapter VIII and **5**]).

In §2 it is shown that the following theorem is a consequence of Theorem 1.

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THEOREM 2. *Let T be a c.n.u. contraction whose defect operator is of Hilbert-Schmidt class. If T is not a weak contraction, then*

$$\{T\}'' = \{\phi(T) \colon \phi \in H^\infty\}.$$

This result was proved in a previous paper [**4**] for $C_{1\cdot}$-contractions. The proof of [**4**, Theorem 2] is useful to our proof of Theorem 1.

2. The following lemma shows that Theorem 2 is an immediate consequence of Theorem 1.

LEMMA 3. *If T is a contraction on a Hilbert space $\mathcal{H}$ whose defect operator is of Hilbert-Schmidt class, and which is not a weak contraction, then there exists an operator W with dense range such that $WT = SW$ or $WT^* = SW$ for some unilateral shift S.*

For $C_{\cdot 0}$-contractions and $C_{1\cdot}$-contractions, Lemma 3 was proved in [**9**] (cf. [**5**, Theorem 2]) and [**4**] respectively, that is, for such a contraction T there exists an operator W with dense range such that $WT = SW$ for a unilateral shift S satisfying $\operatorname{ind} S = \operatorname{ind} T$ (for a semi-Fredholm operator A, $\operatorname{ind} A$ denotes Fredholm index). Note that if T is a $C_{\cdot 0}$-contraction or a $C_{1\cdot}$-contraction whose defect operator is of Hilbert-Schmidt class, then it is a weak contraction if and only if $\operatorname{ind} T = 0$ (see [**5**] for a $C_{\cdot 0}$-contraction).

PROOF OF LEMMA 3. Let

$$T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$$

be the triangulation of T on the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that T_1 is of class $C_{\cdot 1}$ and T_2 of class $C_{\cdot 0}$ (see [**2**, Theorem II.4.1]). Since D_T is of Hilbert-Schmidt class, D_{T_1} is of Hilbert-Schmidt class too, and it follows from the identity $D_{T_1}T_1^* = T_1^* D_{T_1^*}$ and the injectivity of the $C_{1\cdot}$-contraction T_1^* that the selfadjoint operators $D_{T_1^*}$ and $D_{T_1}|\overline{\operatorname{ran} T_1^*}$ are unitarily equivalent, so that $D_{T_1^*}$ is of Hilbert-Schmidt class. If $\operatorname{ind} T_1 \neq 0$, then, as remarked above, there exists an operator W_1 with dense range such that $W_1 T_1^* = SW_1$ for some unilateral shift S. Then the operator W defined by $W = W_1$ on $\mathcal{H}_1$ and $W = 0$ on $\mathcal{H}_2$ satisfies $WT^* = SW$ and has dense range. Therefore let us assume that $\operatorname{ind} T_1 = 0$. This assumption implies that $\sigma(T_1)$ is included in the unit circle. Then, since $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$ and T is not a weak contraction, $\sigma(T_2)$ fills D. Also the fact that D_T and $D_{T_1^*}$ are of Hilbert-Schmidt class implies that T_{12} is of trace class and D_{T_2} is of Hilbert-Schmidt class. Therefore, by the result remarked above, we obtain an operator W_2 with dense range such that $W_2 T_2 = SW_2$ for some unilateral shift S, and the operator $W = [0, W_2]$ which has dense range satisfies $WT = SW$. This completes the proof.

3. In this section we prove Theorem 1. For a c.n.u. contraction we use the functional model of Sz.-Nagy and Foias [**2**].

For a Hilbert space $\mathcal{E}$, let $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ denote the spaces of $\mathcal{E}$-valued, L^2- and H^2-functions on the unit circle, respectively. For two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}'$, let $L^\infty(\mathcal{E}, \mathcal{E}')$ and $H^\infty(\mathcal{E}, \mathcal{E}')$ denote the spaces of operator-valued, L^∞- and H^∞-functions on the unit circle whose values are operators from $\mathcal{E}$ to $\mathcal{E}'$, respectively.

For an operator-function F in $L^\infty(\mathcal{E}, \mathcal{E}')$, we use its multiplication operator from $L^2(\mathcal{E})$ to $L^2(\mathcal{E}')$ which is denoted by the same letter F:

$$(Ff)(e^{it}) = F(e^{it})f(e^{it}) \qquad (f \in L^2(\mathcal{E})).$$

Let T be a c.n.u. contraction, and let $\mathcal{D}_T$ denote the closure of the range of the defect operator D_T. For the characteristic function Θ_T of T, which is a contractive operator-function in $H^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$, set

$$\Delta_T(e^{it}) = (I - \Theta_T(e^{it})^*\Theta_T(e^{it}))^{1/2}.$$

Then the (unitarily equivalent) functional model of T is the operator $S(\Theta_T)$ on the Hilbert space

$$H(\Theta_T) = [H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}] \ominus \{\Theta_T h \oplus \Delta_T h \colon h \in H^2(\mathcal{D}_T)\},$$

defined by

$$S(\Theta_T)(f \oplus g) = P(\chi f \oplus \chi g),$$

where $\chi(e^{it}) = e^{it}$ and P denotes the orthogonal projection of $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}$ onto $H(\Theta_T)$ (cf. [**2**, Chapter VI]).

LEMMA 4. *If T is a c.n.u. contraction and there is an operator W with dense range such that $WT = SW$ for some unilateral shift S, then there is an operator-function Φ in $H^\infty(\mathcal{D}_{T^*}, \mathcal{E})$, where $\mathcal{E}$ is a Hilbert space, that is $*$-inner and outer such that*

$$H^\infty(\mathcal{E}, \mathcal{F})\Phi = \{A \in H^\infty(\mathcal{D}_{T^*}, \mathcal{F}) \colon A\Theta_T = 0\}$$

for any Hilbert space $\mathcal{F}$.

PROOF. Let S be the unilateral shift on $H^2(\mathcal{G})$ where $\mathcal{G}$ is a Hilbert space. By the lifting theorem of Sz.-Nagy and Foias (see [**2**, Theorem II.2.3 or **3**]) there is an operator-function $\Psi \in H^\infty(\mathcal{D}_{T^*}, \mathcal{G})$ such that $\Psi\Theta_T = 0$ and $W = [\Psi, 0]|H(\Theta_T)$, which implies that the shift-invariant subspace $\mathcal{M} = \{f \in H^2(\mathcal{D}_{T^*}) \colon \tilde{\Theta}_T f = 0\}$ (where, for an operator-function A in $H^\infty(\mathcal{F}, \mathcal{F}')$, $\tilde{A}$ is an operator-function in $H^\infty(\mathcal{F}', \mathcal{F})$ defined by $\tilde{A}(\lambda) = A(\overline{\lambda})^*$) contains the nonzero subspace $\tilde{\Psi}H^2(\mathcal{G})$, hence $\mathcal{M} = \tilde{\Phi}H^2(\mathcal{E})$, where $\mathcal{E}$ is a nonzero Hilbert space and Φ is a $*$-inner function in $H^\infty(\mathcal{D}_{T^*}, \mathcal{E})$, that is, $\tilde{\Phi}$ is inner (cf. [**2**, Theorem V.3.3]). Let us show that the operator-function Φ satisfies the required conditions. Since $\Phi\Theta_T = 0$, the inclusion $H^\infty(\mathcal{E}, \mathcal{F})\Phi \subseteq \{A \in H^\infty(\mathcal{D}_{T^*}, \mathcal{F}) \colon A\Theta_T = 0\}$ for any Hilbert space $\mathcal{F}$ is clear. Conversely, take $A \in H^\infty(\mathcal{D}_{T^*}, \mathcal{F})$ such that $A\Theta_T = 0$ or equivalently $\tilde{\Theta}_T\tilde{A} = 0$. Then, since $\tilde{A}H^2(\mathcal{F}) \subseteq \mathcal{M} = \tilde{\Phi}H^2(\mathcal{E})$ and $\tilde{\Phi}$ is inner, it follows that $\tilde{\Phi}^*\tilde{A} \in H^\infty(\mathcal{F}, \mathcal{E})$ and $\tilde{A} = \tilde{\Phi}(\tilde{\Phi}^*\tilde{A})$, hence $A \in H^\infty(\mathcal{E}, \mathcal{F})\Phi$. Next we see that Φ is outer, equivalently $\tilde{\Phi}^*(L^2(\mathcal{D}_{T^*}) \ominus H^2(\mathcal{D}_{T^*}))$ is dense in $L^2(\mathcal{E}) \ominus H^2(\mathcal{E})$. Indeed, if g is orthogonal to $\tilde{\Phi}^*(L^2(\mathcal{D}_{T^*}) \ominus H^2(\mathcal{D}_{T^*}))$, then $\tilde{\Phi}g$ is orthogonal to $L^2(\mathcal{D}_{T^*}) \ominus H^2(\mathcal{D}_{T^*})$, that is, $\tilde{\Phi}g$ is in $H^2(\mathcal{D}_{T^*})$. Therefore it follows from $\tilde{\Theta}_T\tilde{\Phi} = 0$ that $\tilde{\Phi}g \in \mathcal{M} = \tilde{\Phi}H^2(\mathcal{E})$, so that $g \in H^2(\mathcal{E})$ because $\tilde{\Phi}$ is inner.

Let $\mathcal{L}$ denote the algebra of all operators $\hat{X}$ on $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}$ that have the form

$$\hat{X} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix},$$

where $A \in H^\infty(\mathcal{D}_{T^*}, \mathcal{D}_{T^*})$, $B \in L^\infty(\mathcal{D}_{T^*}, \mathcal{D}_T)$ and $C \in L^\infty(\mathcal{D}_T, \mathcal{D}_T)$ such that $BH^2(\mathcal{D}_{T^*}) \subseteq \overline{\Delta_T L^2(\mathcal{D}_T)}$ and $C\overline{\Delta_T L^2(\mathcal{D}_T)} \subseteq \overline{\Delta_T L^2(\mathcal{D}_T)}$, and they satisfy the equality

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} = \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} K$$

for some $K \in H^\infty(\mathcal{D}_T, \mathcal{D}_T)$. By the lifting theorem of Sz.-Nagy and Foias (see [**2**, Theorem II.2.3 or **3**]) the correspondence $\pi\colon \hat{X} \mapsto P\hat{X}|H(\Theta_T)$ maps $\mathcal{L}$ onto the commutant $\{S(\Theta_T)\}'$ and obviously it is linear and multiplicative.

PROOF OF THEOREM 1. The inclusion $\{\phi(T)\colon \phi \in H^\infty\} \subseteq \{T\}''$ is obvious. To see the converse inclusion, take $\hat{X} = [\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}] \in \mathcal{L}$ for which $\pi(\hat{X}) \in \{S(\Theta_T)\}''$, and let us prove that there is $\phi \in H^\infty$ such that $\pi(\hat{X}) = \phi(S(\Theta_T))$. Since $A\Theta_T = \Theta_T K$ for some $K \in H^\infty(\mathcal{D}_T, \mathcal{D}_T)$, it follows from Lemma 4 that there is an operator-function $A_1 \in H^\infty(\mathcal{E}, \mathcal{E})$ such that $\Phi A = A_1 \Phi$, where Φ is the operator-function in Lemma 4. Take any $F \in H^\infty(\mathcal{E}, \mathcal{D}_{T^*})$ and $G \in L^\infty(\mathcal{E}, \mathcal{D}_T)$. Since $\Phi\Theta_T = 0$, the operator

$$\hat{Y} = \begin{bmatrix} F\Phi & 0 \\ \Delta_T G\Phi & 0 \end{bmatrix}$$

belongs to $\mathcal{L}$. Therefore the assumption $\pi(\hat{X}) \in \{S(\Theta_T)\}''$ implies that

$$\pi(\hat{X}\hat{Y} - \hat{Y}\hat{X}) = \pi(\hat{X})\pi(\hat{Y}) - \pi(\hat{Y})\pi(\hat{X}) = 0,$$

and so the operator

$$\hat{X}\hat{Y} - \hat{Y}\hat{X} = \begin{bmatrix} AF\Phi - F\Phi A & 0 \\ BF\Phi + C\Delta_T G\Phi - \Delta_T G\Phi A & 0 \end{bmatrix}$$

maps $H^2(\mathcal{D}_{T^*}) \oplus \{0\}$ into $\{\Theta_T f \oplus \Delta_T f\colon f \in H^2(\mathcal{D}_T)\}$. Then, since $\Phi A = A_1\Phi$ and Φ is outer, we have

$$\begin{bmatrix} AF - FA_1 \\ BF + C\Delta_T G - \Delta_T G A_1 \end{bmatrix} H^2(\mathcal{E}) \subseteq \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} H^2(\mathcal{D}_T),$$

so that $A_1\Phi F - \Phi F A_1 = \Phi(AF - FA_1) = 0$ and $A_1\Phi(\chi^{-n}F) = \Phi(\chi^{-n}F)A_1$ for $n = 1, 2, \ldots$, where $\chi(e^{it}) = e^{it}$. Since the set $\{\chi^{-n}F\colon F \in H^\infty(\mathcal{E}, \mathcal{D}_{T^*})$ and $n = 1, 2, \ldots\}$ is operator-weakly dense in $L^\infty(\mathcal{E}, \mathcal{D}_{T^*})$ and $\Phi L^\infty(\mathcal{E}, \mathcal{D}_{T^*}) = L^\infty(\mathcal{E}, \mathcal{E})$ because Φ is $*$-inner, it follows that $A_1 \in H^\infty(\mathcal{E}, \mathcal{E})$ commutes with all of $L^\infty(\mathcal{E}, \mathcal{E})$, which implies $A_1 = \phi I_{\mathcal{E}}$ for some $\phi \in H^\infty$. Now, set $\hat{Z} = [\begin{smallmatrix} \phi I & 0 \\ 0 & \phi I \end{smallmatrix}]$. Obviously, $\hat{Z} \in \mathcal{L}$ and $\pi(\hat{Z}) = \phi(S(\Theta_T))$. Furthermore, for every $F \in H^\infty(\mathcal{E}, \mathcal{D}_{T^*})$ and $G \in L^\infty(\mathcal{E}, \mathcal{D}_T)$

$$(\hat{X} - \hat{Z}) \begin{bmatrix} F \\ \Delta_T G \end{bmatrix} = \begin{bmatrix} (A - \phi I)F \\ BF + (C - \phi I)\Delta_T G \end{bmatrix} = \begin{bmatrix} AF - FA_1 \\ BF + C\Delta_T G - \Delta_T G A_1 \end{bmatrix},$$

so that

$$(\hat{X} - \hat{Z}) \begin{bmatrix} F \\ \Delta_T G \end{bmatrix} H^2(\mathcal{E}) \subseteq \begin{bmatrix} \Theta_T \\ \Delta_T \end{bmatrix} H^2(\mathcal{D}_T).$$

Then, since obviously $H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}$ is the closed linear span of

$$\{Fh \oplus \Delta_T Gh \colon F \in H^\infty(\mathcal{E}, \mathcal{D}_{T^*}),\ G \in L^\infty(\mathcal{E}, \mathcal{D}_T) \text{ and } h \in H^2(\mathcal{E})\},$$

it follows that

$$(\hat{X} - \hat{Z})(H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}) \subseteq \{\Theta_T f \oplus \Delta_T f \colon f \in H^2(\mathcal{D}_T)\},$$

hence $\pi(\hat{X} - \hat{Z}) = 0$, which proves $\pi(\hat{X}) = \phi(S(\Theta_T))$.

4. Finally we consider a contraction T with a unitary part. Let $T = T_1 \oplus U_a \oplus U_s$, where T_1 is a c.n.u. contraction, U_a is an absolutely continuous unitary operator and U_a is a singular unitary operator. Then $\operatorname{Alg} T = \operatorname{Alg}(T_1 \oplus U_a) \oplus \operatorname{Alg} U_s$ (cf. [**11**]) and this implies $\{T\}'' = \{T_1 \oplus U_a\}'' \oplus \{U_s\}''$. Also the singular unitary operator U_s is reductive [**10**], that is, every U_s-invariant subspace is U_s-reducing, hence it follows from the reflexivity of U_s and the double commutant theorem for von Neumann algebras that $\operatorname{Alg} U_s = \{U_s\}''$ (cf. [**6**]). Thus $\operatorname{Alg} T = \{T\}''$ if and only if $\operatorname{Alg}(T_1 \oplus U_a) = \{T_1 \oplus U_a\}''$. Now we have the following theorem.

THEOREM 5. *If T is a contraction and there is an operator W with dense range such that $WT = SW$ for some unilateral shift S, then T has the bicommutant property.*

In fact, since the absolutely continuous unitary operator U_a is similar to a c.n.u. contraction T_2 (cf. [**1**]), $T_1 \oplus U_a$ is similar to $T_1 \oplus T_2$. Also, since obviously $W = 0$ on the space on which the unitary operator $U_a \oplus U_s$ acts, there is an operator W_1 with dense range such that $W_1 T_1 = S W_1$, hence the c.n.u. contraction $T_1 \oplus T_2$ similar to $T_1 \oplus U_a$ satisfies the assumption in Theorem 1. Thus $T_1 \oplus T_2$ and $T_1 \oplus U_a$ have the bicommutant property, and as remarked above, T has the bicommutant property too.

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