CONTRACTIONS WITH THE BICOMMUTANT PROPERTY

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ABSTRACT. It is shown that if $T$ is a contraction for which there is an operator $W$ with dense range such that $WT = SW$ for some unilateral shift $S$, then $T$ has the bicommutant property, that is, the double commutant of $T$ is the weakly closed algebra generated by $T$ and the identity. As an example of such a contraction we have a contraction $T$ such that $I - T^*T$ is of trace class and the spectrum of $T$ fills the unit disc.

1. A bounded linear operator $T$ on a Hilbert space is said to have the bicommutant property if $\{T\}'' = \text{Alg } T$, where $\{T\}''$ and $\text{Alg } T$ denote the double commutant of $T$ and the weakly closed algebra generated by $T$ and the identity, respectively. Every nonunitary isometry has the bicommutant property [6]. This result was extended by Uchiyama [7 and 8] and Wu [12] to some classes of contractions $T$ whose defect operators $D_T = (I - T^*T)^{1/2}$ are of finite rank. In [7 and 8], Uchiyama proved the bicommutant property for $C_0$-contractions not of class $C_00$ whose defect operators are of finite rank. Subsequently Wu [12] proved this property for $C_x-$contractions not of class $C_xx$ whose defect operators are of finite rank. In this note the above results are extended to contractions whose defect operators are of Hilbert-Schmidt class; indeed, we obtain a more general result.

A contraction is completely nonunitary (c.n.u.) if it has no nontrivial unitary direct summand. Given a c.n.u. contraction $T$, the $H^\infty$-functional calculus of Sz.-Nagy and Foias defines the operator $\phi(T)$ in $\text{Alg } T$ for every $\phi$ in $H^\infty$ (cf. [2, Chapter III]). In this note all Hilbert spaces are assumed to be separable.

Our main result is the following theorem, which we prove in §3.

THEOREM 1. If $T$ is a c.n.u. contraction and there exists an operator $W$ with dense range such that $WT = SW$ for some unilateral shift $S$, then $\{T\}'' = \{\phi(T) : \phi \in H^\infty\}$, and in particular $T$ has the bicommutant property.

A contraction $T$ is called a weak contraction if its defect operator $D_T$ is of Hilbert-Schmidt class and its spectrum $\sigma(T)$ does not fill the open unit disc $D$. A $C_1$-contraction with Hilbert-Schmidt defect operator is a weak contraction if and only if it is of class $C_{11}$, and a $C_0$-contraction with Hilbert-Schmidt defect operator is a weak contraction if and only if it is of class $C_00$ (cf. [2, Chapter VIII and 5]).

Received by the editors February 29, 1984.

1980 Mathematics Subject Classification. Primary 47A45; Secondary 47C05.

Key words and phrases. Contraction, the bicommutant property, double commutant, unilateral shift, Hilbert-Schmidt defect operator.

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Theorem 2. Let $T$ be a c.n.u. contraction whose defect operator is of Hilbert-Schmidt class. If $T$ is not a weak contraction, then

\[ \{T\}' = \{\phi(T) : \phi \in H^\infty\}. \]

This result was proved in a previous paper [4] for $C_1$-contractions. The proof of [4, Theorem 2] is useful to our proof of Theorem 1.

2. The following lemma shows that Theorem 2 is an immediate consequence of Theorem 1.

Lemma 3. If $T$ is a contraction on a Hilbert space $\mathcal{H}$ whose defect operator is of Hilbert-Schmidt class, and which is not a weak contraction, then there exists an operator $W$ with dense range such that $WT = SW$ or $WT^* = SW$ for some unilateral shift $S$.

For $C_0$-contractions and $C_1$-contractions, Lemma 3 was proved in [9] (cf. [5, Theorem 2]) and [4] respectively, that is, for such a contraction $T$ there exists an operator $W$ with dense range such that $W = SW$ for a unilateral shift $S$ satisfying $\text{ind } S = \text{ind } T$ (for a semi-Fredholm operator $A$, $\text{ind } A$ denotes Fredholm index). Note that if $T$ is a $C_0$-contraction or a $C_1$-contraction whose defect operator is of Hilbert-Schmidt class, then it is a weak contraction if and only if $\text{ind } T = 0$ (see [5] for a $C_0$-contraction).

Proof of Lemma 3. Let

\[ T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix} \]

be the triangulation of $T$ on the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $T_1$ is of class $C_1$ and $T_2$ of class $C_0$ (see [2, Theorem II.4.1]). Since $D_T$ is of Hilbert-Schmidt class, $D_{T_1}$ is of Hilbert-Schmidt class too, and it follows from the identity $D_{T_1} T_1^* = T_1^* D_{T_1}$ and the injectivity of the $C_1$-contraction $T_1^*$ that the selfadjoint operators $D T_1^*$ and $D_{T_1}|\text{ran } T_1^*$ are unitarily equivalent, so that $D T_1^*$ is of Hilbert-Schmidt class. If $\text{ind } T_1 \neq 0$, then, as remarked above, there exists an operator $W_1$ with dense range such that $W_1 T_1^* = SW_1$ for some unilateral shift $S$. Then the operator $W$ defined by $W = W_1$ on $\mathcal{H}_1$ and $W = 0$ on $\mathcal{H}_2$ satisfies $WT^* = SW$ and has dense range. Therefore let us assume that $\text{ind } T_1 = 0$. This assumption implies that $\sigma(T_1)$ is included in the unit circle. Then, since $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$ and $T$ is not a weak contraction, $\sigma(T_2)$ fills $D$. Also the fact that $D_T$ and $D_{T_1}$ are of Hilbert-Schmidt class implies that $T_{12}$ is of trace class and $D_{T_2}$ is of Hilbert-Schmidt class. Therefore, by the result remarked above, we obtain an operator $W_2$ with dense range such that $W_2 T_2 = SW_2$ for some unilateral shift $S$, and the operator $W = [0, W_2]$ which has dense range satisfies $WT = SW$. This completes the proof.

3. In this section we prove Theorem 1. For a c.n.u. contraction we use the functional model of Sz.-Nagy and Foias [2].

For a Hilbert space $\mathcal{E}$, let $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ denote the spaces of $\mathcal{E}$-valued, $L^2$- and $H^2$-functions on the unit circle, respectively. For two Hilbert spaces $\mathcal{E}$ and $\mathcal{E}'$, let $L^\infty(\mathcal{E}, \mathcal{E}')$ and $H^\infty(\mathcal{E}, \mathcal{E}')$ denote the spaces of operator-valued, $L^\infty$- and $H^\infty$-functions on the unit circle whose values are operators from $\mathcal{E}$ to $\mathcal{E}'$, respectively.
For an operator-function $F$ in $L^\infty(\mathcal{E}, \mathcal{E}')$, we use its multiplication operator from $L^2(\mathcal{E})$ to $L^2(\mathcal{E}')$ which is denoted by the same letter $F$:

$$(Ff)(e^{it}) = F(e^{it})f(e^{it}) \quad (f \in L^2(\mathcal{E})).$$

Let $T$ be a c.n.u. contraction, and let $D_T$ denote the closure of the range of the defect operator $D_T$. For the characteristic function $\Theta_T$ of $T$, which is a contractive operator-function in $H^\infty(D_T, D_T')$, set

$$\Delta_T(e^{it}) = (I - \Theta_T(e^{it})^*\Theta_T(e^{it}))^{1/2}.$$

Then the (unitarily equivalent) functional model of $T$ is the operator $S(\Theta_T)$ on the Hilbert space

$$H(\Theta_T) = [H^2(D_T) \oplus \Delta_T L^2(D_T)] \oplus \{\Theta_T h + \Delta_T h : h \in H^2(D_T)\},$$

defined by

$$S(\Theta_T)(f \oplus g) = P(\chi f \oplus \chi g),$$

where $\chi(e^{it}) = e^{it}$ and $P$ denotes the orthogonal projection of $H^2(D_T) \oplus \Delta_T L^2(D_T)$ onto $H(\Theta_T)$ (cf. [2, Chapter VI]).

**Lemma 4.** If $T$ is a c.n.u. contraction and there is an operator $W$ with dense range such that $WT = SW$ for some unilateral shift $S$, then there is an operator-function $\Phi$ in $H^\infty(D_T, \mathcal{E})$, where $\mathcal{E}$ is a Hilbert space, that is $*$-inner and outer such that

$$H^\infty(\mathcal{E}, \mathcal{F})\Phi = \{A \in H^\infty(D_T, \mathcal{F}) : A\Theta_T = 0\}$$

for any Hilbert space $\mathcal{F}$.

**Proof.** Let $S$ be the unilateral shift on $H^2(\mathcal{G})$ where $\mathcal{G}$ is a Hilbert space. By the lifting theorem of Sz.-Nagy and Foias (see [2, Theorem II.2.3 or 3]) there is an operator-function $\Psi \in H^\infty(D_T, \mathcal{G})$ such that $\Psi\Theta_T = 0$ and $W = [\Psi, 0]|H(\Theta_T)$, which implies that the shift-invariant subspace $M = \{f \in H^2(D_T) : \Theta_T f = 0\}$ (where, for an operator-function $A$ in $H^\infty(\mathcal{F}, \mathcal{F}')$, $\tilde{A}$ is an operator-function in $H^\infty(\mathcal{F}', \mathcal{F})$ defined by $\tilde{A}(\lambda) = A(\lambda)^*$) contains the nonzero subspace $\tilde{\Psi}H^2(\mathcal{G})$, hence $M = \tilde{\Phi}H^2(\mathcal{E})$, where $\mathcal{E}$ is a nonzero Hilbert space and $\Phi$ is a $*$-inner function in $H^\infty(D_T, \mathcal{E})$, that is, $\Phi$ is inner (cf. [2, Theorem V.3.3]). Let us show that the operator-function $\Phi$ satisfies the required conditions. Since $\Phi\Theta_T = 0$, the inclusion $H^\infty(\mathcal{E}, \mathcal{F})\Phi \subseteq \{A \in H^\infty(D_T, \mathcal{F}) : A\Theta_T = 0\}$ for any Hilbert space $\mathcal{F}$ is clear. Conversely, take $A \in H^\infty(D_T, \mathcal{F})$ such that $A\Theta_T = 0$ or equivalently $\tilde{\Theta}_T\tilde{A} = 0$. Then, since $\tilde{A}H^2(\mathcal{F}) \subseteq M = \tilde{\Phi}H^2(\mathcal{E})$ and $\Phi$ is inner, it follows that $\tilde{\Phi}^*\tilde{A} \in H^\infty(\mathcal{E}, \mathcal{F})$ and $\tilde{A} = \tilde{\Phi}(\tilde{\Phi}^*\tilde{A})$, hence $A \in H^\infty(\mathcal{E}, \mathcal{F})\Phi$. Next we see that $\Phi$ is outer, equivalently $\tilde{\Phi}^*\tilde{L}(H^2(\mathcal{D}))(\tilde{\Phi}^*\tilde{H}^2(\mathcal{D}))$ is dense in $L^2(\mathcal{E})\oplus H^2(\mathcal{E})$. Indeed, if $g$ is orthogonal to $\tilde{\Phi}^*\tilde{L}(H^2(\mathcal{D}))(\tilde{\Phi}^*\tilde{H}^2(\mathcal{D}))$, then $\tilde{\Phi}g$ is orthogonal to $L^2(\mathcal{D})(\tilde{\Phi}^*\tilde{H}^2(\mathcal{D}))$, that is, $\tilde{\Phi}g$ is in $H^2(\mathcal{D})$. Therefore it follows from $\tilde{\Theta}_T\tilde{\Phi} = 0$ that $\tilde{\Phi}g \in M = \tilde{\Phi}H^2(\mathcal{E})$, so that $g \in H^2(\mathcal{E})$ because $\Phi$ is inner.

Let $\mathcal{L}$ denote the algebra of all operators $\tilde{X}$ on $H^2(D_T) \oplus \Delta_T L^2(D_T)$ that have the form

$$\tilde{X} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix},$$
where $A \in H^\infty(\mathcal{D}_{T}, \mathcal{D}_{T})$, $B \in L^\infty(\mathcal{D}_{T'}, \mathcal{D}_{T})$ and $C \in L^\infty(\mathcal{D}_{T}, \mathcal{D}_{T})$ such that $BH^2(\mathcal{D}_{T'}) \subseteq \Delta_T L^2(\mathcal{D}_{T})$ and $C\Delta_T L^2(\mathcal{D}_{T}) \subseteq \Delta_T L^2(\mathcal{D}_{T})$, and they satisfy the equality
\[
\begin{bmatrix}
A & 0 \\
B & C
\end{bmatrix}
\begin{bmatrix}
\Theta_T \\
\Delta_T
\end{bmatrix}
= 
\begin{bmatrix}
\Theta_T \\
\Delta_T
\end{bmatrix}
K
\]
for some $K \in H^\infty(\mathcal{D}_{T}, \mathcal{D}_{T})$. By the lifting theorem of Sz.-Nagy and Foias (see [2, Theorem II.2.3 or 3]) the correspondence $\pi: \hat{X} \mapsto P\hat{X}|H(\Theta_T)$ maps $\mathcal{L}$ onto the commutant $\{S(\Theta_T)\}'$ and obviously it is linear and multiplicative.

**Proof of Theorem 1.** The inclusion $\{\phi(T): \phi \in H^\infty\} \subseteq \{T\}''$ is obvious. To see the converse inclusion, take $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}$ for which $\pi(X) \in \{S(\Theta_T)\}''$, and let us prove that there is $\phi \in H^\infty$ such that $\pi(X) = \phi(S(\Theta_T))$. Since $A\Theta_T = \Theta_T K$ for some $K \in H^\infty(\mathcal{D}_{T}, \mathcal{D}_{T})$, it follows from Lemma 4 that there is an operator-function $A_1 \in H^\infty(\mathcal{E}, \mathcal{E})$ such that $\Phi A = A_1 \Phi$, where $\Phi$ is the operator-function in Lemma 4. Take any $F \in H^\infty(\mathcal{E}, \mathcal{D}_{T'})$ and $G \in L^\infty(\mathcal{E}, \mathcal{D}_{T})$. Since $\Phi \Theta_T = 0$, the operator
\[
\hat{Y} = \begin{bmatrix}
F \Phi & 0 \\
\Delta_T G \Phi & 0
\end{bmatrix}
\]
belongs to $\mathcal{L}$. Therefore the assumption $\pi(X) \in \{S(\Theta_T)\}''$ implies that
\[
\pi(X\hat{X} - \hat{X} X) = \pi(X)\pi(\hat{X}) - \pi(\hat{X})\pi(X) = 0,
\]
and so the operator
\[
X\hat{X} - \hat{X} X = \begin{bmatrix}
AF - F\Phi A & 0 \\
BF + C\Delta_T G \Phi - \Delta_T G \Phi A & 0
\end{bmatrix}
\]
maps $H^2(\mathcal{D}_{T'}) \oplus \{0\}$ into $\{\Theta_T f \oplus \Delta_T f: f \in H^2(\mathcal{D}_{T})\}$. Then, since $\Phi A = A_1 \Phi$ and $\Phi$ is outer, we have
\[
\begin{bmatrix}
AF - FA_1 \\
BF + C\Delta_T G - \Delta_T GA_1
\end{bmatrix}
H^2(\mathcal{E}) \subseteq \begin{bmatrix}
\Theta_T \\
\Delta_T
\end{bmatrix}
H^2(\mathcal{D}_{T'}),
\]
so that $A_1 \Phi F - \Phi FA_1 = \Phi(AF - FA_1) = 0$ and $A_1 \Phi(\chi^n F) = \Phi(\chi^n F) A_1$ for $n = 1, 2, \ldots$, where $\chi(e^{it}) = e^{it}$. Since the set $\{\chi^n F: F \in H^\infty(\mathcal{E}, \mathcal{D}_{T'})\}$ and $n = 1, 2, \ldots$ is operator-weakly dense in $L^\infty(\mathcal{E}, \mathcal{D}_{T'})$ and $\Phi L^\infty(\mathcal{E}, \mathcal{D}_{T'}) = L^\infty(\mathcal{E}, \mathcal{E})$ because $\Phi$ is $*$-inner, it follows that $A_1 \in H^\infty(\mathcal{E}, \mathcal{E})$ commutes with all of $L^\infty(\mathcal{E}, \mathcal{E})$, which implies $A_1 = \phi I_{\mathcal{E}}$ for some $\phi \in H^\infty$. Now, set $\hat{Z} = \begin{bmatrix} \phi I & 0 \\ 0 & \phi I \end{bmatrix}$. Obviously, $\hat{Z} \in \mathcal{L}$ and $\pi(\hat{Z}) = \phi(S(\Theta_T))$. Furthermore, for every $F \in H^\infty(\mathcal{E}, \mathcal{D}_{T'})$ and $G \in L^\infty(\mathcal{E}, \mathcal{D}_{T})$
\[
(\hat{X} - \hat{Z}) \begin{bmatrix}
F \\
\Delta_T G
\end{bmatrix}
= 
\begin{bmatrix}
(A - \phi I)F \\
BF + (C - \phi I)\Delta_T G
\end{bmatrix}
= 
\begin{bmatrix}
AF - FA_1 \\
BF + C\Delta_T G - \Delta_T GA_1
\end{bmatrix},
\]
since
\[
(\hat{X} - \hat{Z}) \begin{bmatrix}
F \\
\Delta_T G
\end{bmatrix}
H^2(\mathcal{E}) \subseteq \begin{bmatrix}
\Theta_T \\
\Delta_T
\end{bmatrix}
H^2(\mathcal{D}_{T'}).
Then, since obviously $H^2(D_{T^*}) \oplus \Delta_T L^2(D_T)$ is the closed linear span of
\[ \{ Fh \oplus \Delta_T Gh : F \in H^\infty(\mathcal{E}, D_{T^*}), \ G \in L^\infty(\mathcal{E}, D_T) \text{ and } h \in H^2(\mathcal{E}) \} , \]
it follows that
\[ (\hat{X} - \hat{Z})(H^2(D_{T^*}) \oplus \Delta_T L^2(D_T)) \subseteq \{ \Theta_T f \oplus \Delta_T f : f \in H^2(D_T) \} , \]
hence $\pi(\hat{X} - \hat{Z}) = 0$, which proves $\pi(\hat{X}) = \phi(S(\Theta_T))$.

4. Finally we consider a contraction $T$ with a unitary part. Let $T = T_1 \oplus U_a \oplus U_s$, where $T_1$ is a c.n.u. contraction, $U_a$ is an absolutely continuous unitary operator and $U_s$ is a singular unitary operator. Then $\text{Alg} T = \text{Alg}(T_1 \oplus U_a) \oplus \text{Alg} U_s$ (cf. [11]) and this implies $\{ T \}'' = \{ T_1 \oplus U_a \}'' \oplus \{ U_s \}''$. Also the singular unitary operator $U_s$ is reductive [10], that is, every $U_s$-invariant subspace is $U_s$-reducing, hence it follows from the reflexivity of $U_s$ and the double commutant theorem for von Neumann algebras that $\text{Alg} U_s = \{ U_s \}''$ (cf. [6]). Thus $\text{Alg} T = \{ T \}''$ if and only if $\text{Alg}(T_1 \oplus U_a) = \{ T_1 \oplus U_a \}''$. Now we have the following theorem.

THEOREM 5. If $T$ is a contraction and there is an operator $W$ with dense range such that $WT = SW$ for some unilateral shift $S$, then $T$ has the bicommutant property.

In fact, since the absolutely continuous unitary operator $U_a$ is similar to a c.n.u. contraction $T_2$ (cf. [1]), $T_1 \oplus U_a$ is similar to $T_1 \oplus T_2$. Also, since obviously $W = 0$ on the space on which the unitary operator $U_a \oplus U_s$ acts, there is an operator $W_1$ with dense range such that $W_1 T_1 = SW_1$, hence the c.n.u. contraction $T_1 \oplus T_2$ similar to $T_1 \oplus U_a$ satisfies the assumption in Theorem 1. Thus $T_1 \oplus T_2$ and $T_1 \oplus U_a$ have the bicommutant property, and as remarked above, $T$ has the bicommutant property too.

REFERENCES

5. K. Takahashi and M. Uchiyama, Every $C_{00}$-contraction with Hilbert Schmidt defect operator is of class $C_0$, J. Operator Theory 10 (1983), 331–335.

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