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A RIGID SPACE WHOSE SQUARE IS THE HILBERT SPACE

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ABSTRACT. We construct a space X with only one autohomeomorphism and the property that the product $X \times X$ is homeomorphic to the separable Hilbert space.

1. Introduction. In [4] J. van Mill constructed a rigid space (i.e., a space with only one autohomeomorphism) whose square was the Hilbert cube. Since then several finite-dimensional rigid spaces with a square that is a manifold have been found; see Ancel and Singh [1] and Ancel, Duvall and Singh [2]. The examples in these papers are constructed by means of CE-decompositions of manifolds. This idea does not work if one wants to factorize the Hilbert space l^2 into rigid spaces (see Mogilski [5], who proved that any absolute retract that is a CE-image of an l^2-manifold is homeomorphic to l^2). Nevertheless, in this note we give a remarkably simple construction of $2^{\aleph_0}$ distinct rigid spaces with squares that are homeomorphic to l^2. It is also worth noting that [1, 2 and 4] use algebraic arguments, whereas our discussion is purely topological.

2. Preliminaries. In this section we have collected a few known facts related to continua. We use Sierpiński's theorem [6] that no continuum can be partitioned into countably many disjoint closed subsets. If x is a point of a space X then the *continuum-component* of x is given by

$$\mathrm{CC}(x, X) = \bigcup \{ C \subset X \mid C \text{ is a continuum that contains } x \}.$$

The space X is called *continuum-connected* if $\mathrm{CC}(x, X) = X$ for some x. Note that Sierpiński's theorem is also valid for continuum-connected spaces.

A closed, continuous mapping is called *monotone* if it is onto and has the property that the preimage of every connected set is also connected. The following lemma is due to Anderson, Curtis and van Mill [3].

LEMMA. *Let B_1 and B_2 be σ-Z-sets in the Hilbert cube Q, and let $f\colon Q \setminus B_1 \to Q \setminus B_2$ be a homeomorphism. Then there exist a compact space M and monotone mappings $\gamma_1, \gamma_2 \colon M \to Q$ such that $\gamma_1^{-1}(B_1) = \gamma_2^{-1}(B_2)$ and $f \circ \gamma_1 | \gamma_1^{-1}(Q \setminus B_1) = \gamma_2 | \gamma_2^{-1}(Q \setminus B_2)$.*

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3. The example. Let the Hilbert cube be represented by $Q = \prod_{i=1}^{\infty}[-1,1]$, and let $(p_i)_{i=1}^{\infty}$ be a sequence in $(0,1)$ converging to 1. Define, for every $i \in \mathbf{N}$, the "shrunken endface in the i-coordinate direction" W_i by

$$W_i = \left\{(x_j)_{j=1}^{\infty} \in Q | x_i = 1 \text{ and } x_j \in [-p_i, p_i] \text{ for } j \neq i\right\},$$

and put $Y = Q \setminus \bigcup_{i=1}^{\infty} W_i$. The space Y was introduced by Anderson, Curtis and van Mill [**3**]. It is homogeneous and has the following interesting property (see [**3**, §3]): if A is a σ-compact subset of Y then $(Y \setminus A) \times (Y \setminus A)$ is homeomorphic to l^2. We construct a σ-compactum in Y such that its complement is rigid.

For every $i \in \mathbf{N}$ let Z_i be an i-pointed star with centre x_i (i.e., Z_i is the disjoint union $\{x_i\} \cup \bigcup_{j=1}^{i} J_{ij}$, where $J_{ij} \cup \{x_i\}$ is homeomorphic to the arc $[0,1]$). It is no problem to imbed the Z_i's pairwise disjointly in Y such that the set $\{x_i | i \in \mathbf{N}\}$ is dense (imbed them, for instance, in the pseudo-interior $\prod_{i=1}^{\infty}(-1,1)$, in which case it is also obvious that every Z_i is a Z-set in Q). Our example X is now given by

$$X = Y \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} J_{ij}.$$

Observe that $P = Q \setminus X$ is a σ-Z-set in Q and the W_i's are disjoint copies of Q. Since $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} J_{ij}$ is σ-compact it suffices to show that X is rigid.

Claim. X is rigid.

PROOF. Note that

$$\{W_i | i \in \mathbf{N}\} \cup \left\{J_{ij} | (i, j) \in \mathbf{N}^2, j \leqslant i\right\}$$

forms a countable, closed covering of P consisting of disjoint continuum-connected sets. According to Sierpiński, this means that it is the collection of continuum-components of P. An analogous argument yields that

$$\mathrm{CC}(x_i, P \cup \{x_i\}) = Z_i \quad \text{for } i \in \mathbf{N}$$

and

$$\mathrm{CC}(x, P \cup \{x\}) = \{x\} \quad \text{for } x \in X \setminus \{x_i | i \in \mathbf{N}\}.$$

This means that x_i is the only point x of X whose continuum-component in $P \cup \{x\}$ contains precisely i continuum-components of P.

Let h be an autohomeomorphism of X. According to the lemma, there exist a compact M and monotone mappings $\gamma_1, \gamma_2 \colon M \to Q$ such that $\gamma_1^{-1}(P) = \gamma_2^{-1}(P)$ and $h \circ \gamma_1 | \gamma_1^{-1}(X) = \gamma_2 | \gamma_2^{-1}(X)$. Observe that if $A \subset B \subset Q$ then A is a continuum-component of B iff $\gamma_k^{-1}(A)$ is a continuum-component of $\gamma_k^{-1}(B)$. Let x be a point of X. Since $\gamma_1^{-1}(\{x\}) = \gamma_2^{-1}(\{h(x)\}) \neq \varnothing$, we have

$$\gamma_1^{-1}(\mathrm{CC}(x, P \cup \{x\})) = \gamma_2^{-1}(\mathrm{CC}(h(x), P \cup \{h(x)\})).$$

Call this last set C and note that $\mathrm{CC}(x, P \cup \{x\})$ contains as many continuum-components of P as C contains continuum-components of $\gamma_1^{-1}(P) = \gamma_2^{-1}(P)$. Since the same is true for $h(x)$, we have that $\mathrm{CC}(x, P \cup \{x\})$ contains precisely the same number of continuum-components of P as $\mathrm{CC}(h(x), P \cup \{h(x)\})$. In view of the above remark, we may conclude that h fixes the dense set $\{x_i | i \in \mathbf{N}\}$. This means that h is the identity and, hence, X is rigid.

REMARK. Note that this method yields $2^{\aleph_0}$ topologically distinct rigid spaces with squares homeomorphic to l^2.

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