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A CHARACTERIZATION OF CLARKE'S STRICT TANGENT CONE VIA NONLINEAR SEMIGROUPS

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ABSTRACT. Clarke's strict tangent cone $T_X^\uparrow(a)$ at $a \in X$ to a closed subset of a Banach space E is shown to contain the limit inferior of tangent cones $T_X(x)$ to X at x as $x \to a$, $x \in X$. Several characterizations of $T_X^\uparrow(a)$ are presented. As a consequence various tangential and subtangential conditions for continuous vector fields on X are shown to be equivalent.

It is well known that invariance results for dynamical systems on closed subsets of Banach spaces are linked with tangency conditons (see, for instance, [**4**–**7**, **10**, **12**, **16**, **17**] and their references). On the other hand, some results on tangent cones can be deduced from the study of dynamical systems (see, for instance, [**12**, **16**]). In this note we follow this second streamline in order to give characterizations of the strict tangent cone (or Clarke's tangent cone) to a closed subset in a Banach space. These characterizations were given in [**13**] under restrictive assumptions, valid for instance in the finite dimensional case. Other studies containing some of these implications can be found in [**1**, **4**, **7**–**9**] for instance.

1. Characterizations of the strict tangent cone. Given a subset X of a Banach space E we denote by $d_X(e)$ the distance of $e \in E$ to X: $d_X(e) = \inf\{d(e, x)\colon x \in X\}$. We set $B(a, r) = \{x \in X\colon d(a, x) \leqslant r\}$.

Let us recall that the classical *tangent cone* (also called contingent cone) at $a \in X$ to X is the set $T_X(a)$ (also denoted elsewhere $T(X, a)$ or T_aX) of vectors $v \in E$ such that $d'_X(a, v) \leqslant 0$, where

$$d'_X(a, v) =: \liminf_{(t, w) \to (0_+, v)} t^{-1}(d_X(a + tw) - d_X(a))$$
$$= \liminf_{t \to 0_+} t^{-1}(d_X(a + tv) - d_X(a)).$$

The strict *tangent cone* (or Clarke's tangent cone) is the set $T_X^\uparrow(a)$ of vectors $v \in E$ such that $d_X^\uparrow(a, v) \leqslant 0$, where

$$d_X^\uparrow(a, v) = \limsup_{(t, e) \to (0_+, a)} t^{-1}(d_X(e + tv) - d_X(e)).$$

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Both cones have been extensively used in optimization and nonlinear analysis. The first one gives a closer approximation to the set X around a than the second one; but it is not necessarily convex. The second one does not necessarily increase with X. Some algebraic links between the two cones have been delineated in [**11** and **14**]. Here we focus our attention on the relationships between the two cones obtained by taking limits as $x \to a$, $x \in X$.

The following result was shown in [**2**] to be a consequence of a general ordering principle (see also [**6**] for related material).

PROPOSITION 1. *Let F be a closed subset of a Banach space E, let $c \in \mathbf{R}_+$, $\omega \in]0, +\infty[$ and let S be a continuous semigroup on E such that*
(a) $d(S(t)x, S(t)y) \leqslant e^{\omega t} d(x, y)$ *for each* $(t, x, y) \in \mathbf{R}_+ \times E \times E$,
(b) $\liminf_{t \to 0_+} t^{-1} d(S(t)x, F) \leqslant c$ *for each* $x \in F$.
Then $d(S(t)z, F) \leqslant e^{\omega t} d(z, F) + c\omega^{-1}(e^{\omega t} - 1)$ *for each* $t \in \mathbf{R}_+$ *and each* $z \in E$.

In the following theorem the restrictive assumptions made in [**13**] are dropped.

THEOREM 1. *Let a be a point of a closed subset X of a Banach space E. For any $v \in E$ the following assertions are equivalent*:
(a) $v \in T_X^{\uparrow}(a)$;
(b) $\limsup_{e \to a} \limsup_{t \to 0_+} t^{-1}(d_X(e + tv) - d_X(e)) \leqslant 0$;
(b′) $\lim_{x \to a,\, x \in X} \limsup_{t \to 0_+} t^{-1} d_X(x + tv) = 0$;
(c) $\limsup_{e \to a} \liminf_{t \to 0_+} t^{-1}(d_X(e + tv) - d_X(e)) \leqslant 0$;
(c′) $\lim_{x \to a,\, x \in X} \liminf_{t \to 0_+} t^{-1} d_X(x + tv) = 0$.

PROOF. The implications (b) ⇒ (b′), (c) ⇒ (c′), (b) ⇒ (c) and (b′) ⇒ (c′) are obvious; the implication (a) ⇒ (b) follows from the following inequality in which $q(t, e) = t^{-1}(d_X(e + tv) - d_X(e))$:

$$\inf_{\alpha > 0} \sup_{e \in B(a,\alpha)} \inf_{\beta > 0} \sup_{t \in]0,\beta[} q(t, e) \leqslant \inf_{\alpha > 0} \inf_{\beta > 0} \sup_{e \in B(a,\alpha)} \sup_{t \in]0,\beta[} q(t, e).$$

Thus it suffices to show that (c′) implies (a). There is no loss of generality in supposing $\|v\| \leqslant 1$. Let $\varepsilon > 0$ be given; we will show that

$$\limsup_{(t,e) \to (0_+, a)} t^{-1}\big(d_X(e + tv) - d_X(e)\big) \leqslant 3\varepsilon.$$

Using (c′) we can find $\delta > 0$ such that $\liminf_{t \to 0} t^{-1} d_X(x + tv) \leqslant \varepsilon$ for each $x \in X \cap B$, where $B = B(a, 3\delta)$ is the closed ball with center a and radius 3δ. Let $F = X \cap B$, let $\lambda(e) = \delta^{-1}\min(\delta, d(e, B^c))$ with $B^c = E \setminus B$, and let S be the semigroup generated by the lipschitzian vector field V: $E \to E$ given by $V(e) = \lambda(e)v$. As is well known, the flow of V is defined on $\mathbf{R} \times E$ as V is globally lipschitzian (with Lipschitz constant $\omega = \delta^{-1}$) so that S is well defined and satisfies condition (a) of Proposition 1 (cf. [**3**] for instance). Let us check condition (b): for each $x \in F$

$$t^{-1} d(S(t)x, F) \leqslant t^{-1} d(S(t)x, x + tV(x)) + t^{-1} d(x + tV(x), F).$$

As $\lim_{t\to 0_+} t^{-1}(S(t)x - x) = V(x)$, the first term of the right-hand side has limit 0; the second one has limit inferior $\lambda(x)\liminf_{s\to 0_+} s^{-1}d(x + sv, X) \leqslant \varepsilon$ when $d(a, x) < 3\delta$, $x \in X$ and limit 0 when $d(a, x) = 3\delta$, $x \in X$. Thus condition (b) is satisfied with $c = \varepsilon$.

For each $z \in B(a, \delta)$ we have $d_X(z) \leqslant d(z, a) \leqslant \delta$, and any $x \in X$ such that $d(z, x) \leqslant 2\delta$ is in $B(a, 3\delta)$, so that $d_X(z) = d(z, F)$. On the other hand, as $F \subset X$ we have $d_X(z + tv) \leqslant d(z + tv, F)$ for any $t \in \mathbf{R}_+$. Let $\alpha \in]0, \delta]$ be so small that $2\omega\alpha \leqslant \varepsilon$, $t^{-1}(e^{\omega t} - 1) \leqslant 2\omega$ for $t \in]0, \alpha]$. Then for $t \in]0, \alpha]$, $z \in B(a, \alpha)$ we have $S(t)z = z + tv$ and

$$\begin{aligned} t^{-1}(d_X(z + tv) - d_X(z)) &\leqslant t^{-1}(d(z + tv, F) - d(z, F)) \\ &\leqslant t^{-1}(e^{\omega t} - 1)(d(z, F) + \omega^{-1}\varepsilon) \\ &\leqslant 2\omega(\alpha + \omega^{-1}\varepsilon) \leqslant 3\varepsilon. \quad \square \end{aligned}$$

2. Some consequences. Given $x \in X$ and $v \in E$ we define the *contingency coefficient* (or tangency coefficient) of v at x with respect to X as

$$k_X(x, v) = \liminf_{t \to 0_+} t^{-1}d_X(x + tv).$$

Let us set

$$T_X^\varepsilon(x) = \{v \in E: k_X(x, v) \leqslant \varepsilon\|v\|\},$$

so that $T_X^\varepsilon(x)$ is a cone and $T_X(x) = \bigcap_{\varepsilon > 0} T_X^\varepsilon(x)$.

COROLLARY 1. *For any closed subset X of a Banach space and any $a \in X$ one has*

$$\liminf_{\substack{(x,\varepsilon)\to(a,0)\\ x\in X,\ \varepsilon>0}} T_X^\varepsilon(x) \subset T_X^\uparrow(a).$$

PROOF. Suppose v belongs to the left-hand side of this inclusion. Then for any $\alpha > 0$ there exists $\beta > 0$ such that for any $x \in X \cap B(a, \beta)$ and any $\varepsilon \in]0, \beta[$ there exists $v' \in T_X^\varepsilon(x) \cap B(v, \alpha)$. Thus

$$k_X(x, v) \leqslant k_X(x, v') + d(v', v) \leqslant \varepsilon(\|v\| + \alpha) + \alpha.$$

As $\varepsilon \in]0, \beta[$ is arbitrary, we get $k_X(x, v) \leqslant \alpha$ for $x \in X \cap B(a, \beta)$ whence $\lim_{x\to a;\, x\in X} k_X(x, v) = 0$ and $v \in T_X^\uparrow(a)$ by Theorem 1. $\square$

COROLLARY 2. *For any closed subset X of a Banach space and any $a \in X$ one has*

$$\liminf_{\substack{x\to a\\ x\in X}} T_X(x) \subset T_X^\uparrow(a).$$

This follows from the preceding corollary and the fact that $T_X(x) \subset T_X^\varepsilon(x)$ for any $x \in X$ and any $\varepsilon > 0$.

COROLLARY 3. *Let X be a closed subset of a Banach space E and let $a \in E$. Suppose $v \in E$ is such that for any subset A of X with a in its closure one has $v \in \limsup_{x\to a,\, x\in A} T_X(x)$. Then $v \in T_X^\uparrow(a)$.*

This follows from the fact that for any relation $F\colon X \to E$ one has

$$\liminf_{x \to a} F(x) = \bigcap_{A \in \mathscr{A}} \limsup_{\substack{x \to a \\ x \in A}} F(x),$$

where $\mathscr{A}$ is the family of subsets A of X whose closure contains a.

COROLLARY 4. *Let $V\colon X \to E$ be a continuous vector field on a closed subset X of a Banach space E. Then the following assertions are equivalent*:

(a) *for each* $x \in X$, $V(x) \in T_X^{\uparrow}(x)$;

(b) *for each* $x \in X$, $V(x) \in T_X(x)$;

(c) *for each* $x \in X$, $\lim_{t \to 0_+} t^{-1} d_X(x + tV(x)) = 0$;

(d) *for each* $x \in X$, $\liminf_{t \to 0_+} t^{-1} d_X(x + tV(x)) = 0$.

PROOF. The implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Leftrightarrow$ (b) are obvious. Suppose (b) is satisfied. Then for each $x \in X$ we have $V(x) = \lim_{y \to x,\, y \in X} V(y)$, hence $V(x) \in \liminf_{y \to x,\, y \in X} T_X(y) \subset T_X^{\uparrow}(x)$, hence (a) holds true. $\square$

ADDED IN PROOF. Since the present paper has been submitted for publication, the inclusion of Corollary 2 has been proved by different methods and in an independent way in references [**18** and **19**] below. Both references contain counterexamples showing that the inclusion may be strict.

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