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A NOTE ON THE SET OF PERIODS FOR CONTINUOUS MAPS OF THE CIRCLE WHICH HAVE DEGREE ONE

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ABSTRACT. The main result of this paper is to complete Misiurewicz's characterization of the set of periods of a continuous map f of the circle with degree one (which depends on the rotation interval of f). As a corollary we obtain a kind of perturbation theorem for maps of the circle of degree one, and a new algorithm to compute the set of periods when the rotation interval is known.

Also, for maps of degree one which have a fixed point, we describe the relationship between the characterizations of the set of periods of Misiurewicz and Block.

1. Notation. We denote by N, Z, Q and R, as usual, the set of positive integers, integers, rational and real numbers, respectively.

Let S^1 be the circle and $C_1(S^1)$ be the set of continuous maps from the circle into itself of degree one. For a map $f \in C_1(S^1)$, $P(f)$ denotes the set of periods of f (from now on, by period of a periodic point, we will mean the least period of this point).

We consider Sarkovskii's ordering $\to$ on N, defined as follows

$$3 \to 5 \to 7 \to 9 \to \cdots \to 2 \cdot 3 \to 2 \cdot 5 \to 2 \cdot 7 \to 2 \cdot 9 \to \cdots \to 4 \cdot 3 \to 4 \cdot 5 \to 4 \cdot 7 \to \cdots \to \cdots \to 16 \to 8 \to 4 \to 2 \to 1.$$

For every $s \in N$ we denote by S_s the set $\{n \in N: s \to n\} \cup \{s\}$. Also we define $S_{2^\infty} = \{1, 2, 4, \ldots, 2^n, \ldots\}$. Similarly, for every $b \in N$ we denote by B_b the set $\{n \in N: b \leqslant n\}$, and we write $B_\infty = \varnothing$.

Let $f \in C_1(S^1)$ and let F be a lifting of f. If x is a periodic point of f of period n and X is a real number which satisfies that $\exp(2\pi i X) = x$, then we have $F^n(X) = X + k$ for some $k \in Z$. We shall call the number k/n the rotation number of x and denote it by $\rho(x)$ or $\rho_F(x)$. We denote by $L(f)$ or $L_f(f)$ the set of all rotation numbers of periodic points of f. The following statements are known (see **[BGMY** and **M]**).

(1) $\rho(x)$ does not depend on the choice of X.
(2) If $F' = F + m$, then $\rho_{F'}(x) = \rho_F(x) + m$.
(3) $\rho_{F^m}(x) = m\rho_F(x)$.
(4) If $a < b < c$, $a, c \in L(f)$ and $b \in Q$, then $b \in L(f)$.
(5) $L(f) \cap Z \neq \varnothing$ if and only if $1 \in P(f)$.

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(6) If $a_m \in L(f)$ for $m = 1, 2, 3, \ldots$ and $a = \lim_{n \to \infty} a_m \in Q$, then $a \in L(f)$.

(7) If f has no periodic points, then the limit $\lim_{n \to \infty} (F^n(X) - X)/n$ exists for all $X \in R$, is independent on X and is irrational.

From the above statements we can write $L(f) = [c, d] \cap Q$ for some $c, d \in R$. That is, $L(f)$ is a closed interval (possibly degenerated to one point) on Q and, from now on, we shall call it the rotation interval. If f has no periodic points, we take $c = d = \lim_{n \to \infty} (F^n(X) - X)/n$.

Let j, m be two integers relatively prime; we write $(j, m) = 1$.

Let $f \in C_1(S^1)$ with $L_F(f) = [c, d] \cap Q$. If $c \in Q$, $c = j/m$ and $(j, m) = 1$ (if c is an integer we take $j = c$ and $m = 1$), then we consider the continuous map of the real line $F^m - j$. By Sarkovskii's Theorem (see [**Sa**, **St**, **BGMY**]) there exists $s_c \in N \cup \{2^\infty\}$ such that S_{s_c} is the set of periods of $F^m - j$. In a similar way we define s_d when $d \in Q$.

REMARK. It is not difficult to show that s_c and s_d do not depend on the choice of F.

Let $f \in C_1(S^1)$ with $L(f) = [c, d] \cap Q$. We define the set $S(c)$ as follows:

$$S(c) = \begin{cases} \varnothing & \text{if } c \notin Q. \\ \{m \cdot n : n \in S_{s_c}\} & \text{if } c = j/m \text{ with } (j, m) = 1. \end{cases}$$

Similarly, we define $S(d)$. Also, $M(c, d)$ will denote the set $\{n \in N: c < k/n < d,$ for some integer $k\}$. Obviously $M(c, d) = \varnothing$ if and only if $c = d$.

2. Statement of the results. The next result gives Misiurewicz's characterization of the set of periods of a map $f \in C_1(S^1)$ (see [**M**]).

THEOREM 1. *Let $f \in C_1(S^1)$ and $L(f) = [c, d] \cap Q$. Then $P(f) = S(c) \cup M(c, d) \cup S(d)$. Conversely, for every set A of the above form, there exists a map $f \in C_1(S^1)$ of class C^∞, such that $P(f) = A$.*

The set $S(c)$ is characterized by the period $m \cdot s_c$ and Sarkovskii's ordering, if $c = m/j$ with $(m, j) = 1$. That is, to compute $S(c)$ for a given map $f \in C_1(S^1)$, it is enough to compute s_c. In a similar way, we shall characterize the set $M(c, d)$ depending on the usual ordering and on a finite number of periods. Also we shall give an algorithm which computes the set $M(c, d)$ and, in particular, the above finite set of periods. From this point of view, the following two propositions are given in [**M**]. The first one gives another algorithm to compute the set $M(c, d)$, and the second one is a partial characterization of this set.

In what follows we denote by $E: R \to Z$ the integer part function.

PROPOSITION 2. *Let $[c, d]$ be an interval with $c \neq d$. Then the following hold.*

(1) *If $n \geqslant E(1/(d - c)) + 1$, then $n \in M(c, d)$.*

(2) *$n \in M(c, d)$ if and only if $k/d < n < k/c$ for some $k = 1, 2, 3, \ldots$.*

PROPOSITION 3. *Let $[c, d]$ be an interval with $c \neq d$. Then the following hold.*

(1) *If $(c, d) \cap Z \neq \varnothing$, then $M(c, d) = N$.*

(2) *If $(c, d) \cap Z = \varnothing$ and $\{c, d\} \cap Z \neq \varnothing$, then $M(c, d) = B_b$, where $b = E(1/(d - c)) + 1$.*

(3) *If* $[c, d] \cap Z = \varnothing$, $c = j/m$ *and* $d = p/q$ *with* $(m, q) = 1$, *then* $n \in M(j/m, p/q)$ *if and only if there exists* $r, s \in N$ *such that* $n = (mr + sq)/(mp - jq)$.

The following theorem is our main result, which gives the structure of the set $M(c, d)$, and completes the characterization given in Proposition 3.

THEOREM A. *Let* $[c, d]$ *be an interval with* $c \neq d$. *Then we have*:

(1) *There exist* $b \in N$ *and* H, *a finite subset of* N, *such that*

$$M(c, d) = B_b \cup \left(\bigcup_{h \in H} \{ h \cdot n : n \in N \} \right).$$

(2) *If* $[c, d] \cap Z = \varnothing$, *then there exist two rational numbers* j/m *and* p/q *with* $(j, m) = (p, q) = (m, q) = 1$ *such that* $M(c, d) = M(j/m, p/q)$.

The proof of Theorem A(2) is constructive. Then, together with Proposition 3 and Proposition 2(1) it gives a new algorithm to compute the set $M(c, d)$ (see Algorithm D in §3).

From the point of view of the applicability, the interest of having Algorithm D is given by the fact that the computation of $M(c, d)$ can be made essentially by using integer arithmetic, while this is not the case for the algorithm obtained from Proposition 2.

From Theorem A we obtain the following result which is a kind of perturbation theorem (in fact statement (1) can also be obtained from Proposition 2(1) and Theorem 1). Let $f \in C_1(S^1)$ and $L(f) = [c, d] \cap Q$. Assume that either, if $p/q \in \{c, d\}$ with $(p, q) = 1$, then $q \in M(c, d)$ or $\{c, d\} \cap Q = \varnothing$. Then we say that f is *nondegenerate*.

COROLLARY B. *Let* $f \in C_1(S^1)$ *with* $L(f) = [c, d] \cap Q$ *and* $c \neq d$. *Then the following hold.*

(1) *For every map* $f' \in C_1(S^1)$ *with* $L(f') = [c', d'] \cap Q$, *where* $c' \neq d'$, *the change of periods between* $P(f)$ *and* $P(f')$ *occurs in a finite number. That is, there exists an* $n \in N$ *such that* $B_n \subset P(f) \cap P(f')$.

(2) *If* $[c, d] \cap Z = \varnothing$ *there exist* $\varepsilon_1, \varepsilon_2 \geqslant 0$ *and* $\delta_1, \delta_2 > 0$ *such that, for every map* $f' \in C_1(S^1)$ *with* $L(f') = [c', d'] \cap Q$, *where* $c' \in [c - \varepsilon_1, c + \delta_1]$ *and* $d' \in [d - \delta_2, d + \varepsilon_2]$, *we have* $M(c, d) = M(c', d')$. *Moreover, if* f *is nondegenerate and* f' *is sufficiently close of* f, *then* $M(c, d) = M(c', d')$, *where* $[c', d'] \cap Q = L(f')$.

Block's classification of the set of periods of a continuous map of the circle which have a fixed point is given in the following theorem (see **[B1]**).

THEOREM 4. *If* f *is a continuous map of the circle into itself such that* $1 \in P(f)$, *then there exist* $s \in N \cup \{2^\infty\}$ *and* $b \in N \cup \{\infty\}$ *such that* $P(f) = S_s \cup B_b$. *Conversely, for every* $s \in N \cup \{2^\infty\}$ *and* $b \in N \cup \{\infty\}$ *there is a continuous map* f *of the circle into itself such that* $P(f) = S_s \cup B_b$.

Theorems 1 and 4 both work for a map $f \in C_1(S^1)$ such that $1 \in P(f)$. The following result studies the relation between the above two characterizations of the set $P(f)$. In fact, we obtain a formula to write $P(f)$ as $S_s \cup B_b$ depending on $L(f)$, s_c and s_d.

THEOREM C. *Let* $f \in C_1(S^1)$ *with* $1 \in P(f)$ *and* $L(f) = [c, d] \cap Q$. *Then* $L(f) \cap Z \neq \varnothing$ *and the following two statements hold.*

(1) *If* $(c, d) \cap Z \neq \varnothing$ *or* $d - c = 1$, *then* $P(f) = S_3 = N$.

(2) *Suppose that* $\{c, d\} \cap Z \neq \varnothing$ *and* $d - c < 1$. *If* $d - c \in [1/b, 1/(b-1))$, *then we have* $P(f) = S_s \cup B_b$, *where* $s = s_c$ *if* $c \in Z$ *or* $s = s_d$ *if* $d \in Z$.

Now, the natural question is: What about the converse problem? That is, given $P(f)$, what can one say about $L(f)$? If we study this problem we shall find that, by using Theorem C and the standard techniques, it is possible to obtain information about $L(f)$ in a nondifficult—but complicated—way. Unfortunately, this computation will be very rough, specially in the case $P(f) = N$.

3. Proof of the results.

PROOF OF THEOREM A. (1) By Proposition 2, $n \in M(c, d)$ for every $n \in N$ such that $n \geqslant E(1/(d-c)) + 1$. We take $b = \min_{\leqslant}\{n \in M(c, d)\colon B_n \subset M(c, d)\}$. Let H be the set of $n \in M(c, d)$ such that $n < b$ and n does not belong to the set $\{m \cdot k\colon k \in B_2 \text{ and } m \in M(c, d)\}$. Note that H is a finite set. From the definition of $M(c, d)$ it follows that if $n \in M(c, d)$, then $\{n \cdot m\colon m \in N\} \subset M(c, d)$. Hence (1) of Theorem A follows.

(2) Taking the appropriate lifting of f, we can suppose that $[c, d] \subset (0, 1)$. Let $e = E(1/(d-c)) + 1$, and let $0 = r_0 < r_1 < \cdots < r_n = 1$ be all the rational numbers with denominators less than $e + 1$. Note that this set is finite. By Proposition 2, $B_e \subset M(c, d)$. Then $c \in [r_i, r_{i+1})$ and $d \in (r_k, r_{k+1}]$ for some i and k such that $k \geqslant i + 1$. Since $(d - c)^{-1} < e$, we can choose $c' \in [r_i, r_{i+1})$ and $d' \in (r_k, r_{k+1}]$ such that $[c', d'] \subset (0, 1)$ and $(d' - c')^{-1} < e$. We claim that $M(c, d) = M(c', d')$. Since the rationals $\{r_j\colon j = i+1, i+2, \ldots, k\} \subset (c', d')$, we have $(0, e] \cap M(c, d) = (0, e] \cap M(c', d')$. Moreover, since $E(1/(d' - c')) + 1 \leqslant e$ and by Proposition 2, it follows that $[e, +\infty) \cap M(c, d) = [e, +\infty) \cap M(c', d')$. Hence the claim is proved.

Let $t = [e(d-c) - 1]/(2e)$. Now, we shall see that $t > 0$. Since $[c, d] \subset (0, 1)$, there is $m \in N$ such that $d - c \in (1/m, 1/(m-1)]$. Therefore $e = m$ and $e(d-c) - 1 > m(1/m) - 1 = 0$. Hence $t > 0$. Let $c_1 = \min\{c + t, r_{i+1}\}$ and $d_1 = \max\{d - t, r_k\}$. Then the intervals $[r_i, c_1)$ and $(d_1, r_{k+1}]$ have positive length.

We shall show that $1/(d' - c') < e$ for every $d' \in (d_1, r_{k+1}]$ and $c' \in [r_i, c_1)$. It is clear that $d' - c' > d - c - 2t$. Since $2t = d - c - 1/e < d - c$, we have that $d - c - 2t > 0$. In short, it follows that $1/(d' - c') < 1/(d - c - 2t) = e$. Therefore, by the above claim, for every $c' \in [r_i, c_1)$, $d' \in (d_1, r_{k+1}]$ and $[c', d'] \subset (0, 1)$ we have $M(c, d) = M(c', d')$.

Now, we pick up $j/m \in [r_i, c_1)$ and $q' \in N$ such that $(m, j) = 1$, and $(m, q') = 1$, and $1/q' < r_{k+1} - d_1$. Then there is $p' \in N$ such that $p'/q' \in (d_1, r_{k+1}]$. Let $p/q = p'/q'$ with $(p, q) = 1$. Hence $(m, q) = 1$ and $M(c, d) = M(j/m, p/q)$. Q.E.D.

ALGORITHM D. *Let* $[c, d]$ *be an interval with* $c \neq d$ *and* $[c, d] \cap Z = \varnothing$.

(1) *Let* $E(c) = r_0 < r_1 < \cdots < r_n = E(c) + 1$ *be all the rational numbers with denominators less than* $E(1/(d-c)) + 2$.

(2) *There exist i and k with* $k > i$ *such that* $c \in [r_i, r_{i+1})$ *and* $d \in (r_k, r_{k+1}]$.

(3) *Let* $t = (e(d-c)-1)/(2e)$ *and* j/m *such that* $(m, j) = 1$ *and* $r_i \leqslant j/m < \min\{c+t, r_{i+1}\}$, *where* $e = E(1/(d-c)) + 1$.

(4) *Let* $q' \in N$ *such that* $(m, q') = 1$ *and* $1/q' < r_{k+1} - \max\{d-t, r_k\}$. *There is* $p' \in N$ *such that* $\max\{d-t, r_k\} < p'/q' \leqslant r_{k+1}$. *Let* $p/q = p'/q'$, *with* $(p, q) = 1$.

Then $M(c, d) = M(j/m, p/q)$ *with* $(m, j) = (p, q) = (m, q) = 1$.

We shall use the following result in the proof of Corollary B.

LEMMA 5 (SEE [**NPT** AND **M**]). *Let* $f \in C_1(S^1)$ *with* $L(f) = [c, d] \cap Q$. *Then the functions* $c = c(f)$ *and* $d = d(f)$ *depend continuously on* f *(with the topology of uniform convergence) taking the appropriate lifting.*

PROOF OF COROLLARY B. (1) is immediate from (1) of Theorem A, and (2) follows easily from the proof of Theorem A(2) and Lemma 5. Q.E.D.

The following lemma will be used in the proof of Theorem C.

LEMMA 6. *Let* $f \in C_1(S^1)$ *such that* $1 \in P(f)$ *and* $L(f) = [c, d] \cap Q$ *with* $[c, d] \subset [0, 1]$. *Then* $c = 0$ *or* $d = 1$ *and the following statements hold.*

(1) *If* $c = 0$ *and* $m \geqslant 2$, *then* $d \in [1/m, 1/(m-1))$ *if and only if* $M(0, d) \cup S(d) = B_m$.

(2) *If* $d = 1$ *and* $m \geqslant 2$, *then* $c \in ((m-2)/(m-1), (m-1)/m]$ *if and only if* $S(d) \cup M(c, 1) = B_m$.

PROOF. From the definition of rotation number and since $1 \in P(f)$, we have that $c = 0$ or $d = 1$. To prove (1) we shall consider two cases. First, we suppose $d = 1/m$. By (2) of Proposition 3, we obtain $M(0, d) = B_{m+1}$. Since $d = 1/m$ we have that $m \in S(d) = \{m \cdot n \colon n \in S_{s_d}\} \subset B_m$. Then, $M(0, d) \cup S(d) = B_m$. Now, we assume $d \in (1/m, 1/(m-1))$. By Proposition 3(2), we have $M(0, d) = B_m$. Hence, we must prove that $S(d) \subset B_m$. If $d \notin Q$ this is obvious. If $d = p/q$ with $(p, q) = 1$, then $q > m$. Therefore, $S(d) = \{q \cdot n \colon n \in S_{s_d}\} \subset B_m$.

Conversely, if $c = 0$ and $M(0, d) \cup S(d) = B_m$, we have $d \leqslant 1/(m-1)$ (otherwise, by Proposition 3(2), $M(0, d) = B_b$ with $b \leqslant m-1$). If $d = 1/(m-1)$ then $m - 1 \in S(d)$ and this is not possible. Therefore $d < 1/(m-1)$. On the other hand, suppose that $d < 1/m$. Then $m \notin M(0, d)$. If $d \notin Q$ then $S(d) = \varnothing$ and $m \notin M(0, d) \cup S(d)$; this is a contradiction. If $d \in Q$, then $d = p/q$ with $(p, q) = 1$ and $q \geqslant m + 1$ or $d = 0$.

Now, if $d \neq 0$ we have $S(d) \subset B_{m+1}$ and $m \notin M(0, d) \cup S(d)$, and this is also a contradiction. If $d = 0$ we have that $M(0, d) = \varnothing$ and then $S(d) = B_m$. But, for every $s_d \in N$ we have that $1 \in S(d) = B_m$, and this is not possible. In short, we have $d \in [1/m, 1/(m-1))$. The proof of (2) is similar. Q.E.D.

PROOF OF THEOREM C. (1) follows obviously from Proposition 3(1), (2). By taking the appropriate lifting we can suppose that $L(f) \subset [0, 1]$. That is, one of the following two statements hold:

(i) If $c \in Z$, then $d - c \in [1/m, 1/(m-1))$ iff $d \in [1/m, 1/(m-1))$.

(ii) If $d \in Z$, then $d - c \in [1/m, 1/(m-1))$ iff $c \in ((m-2)/(m-1), (m-1)/m]$.

Thus, (2) follows from Lemma 6 and Theorem 1. Q.E.D.

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