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A NOTE ON SKEW-HOPF FIBRATIONS

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ABSTRACT. Each great circle fibration of the unit 3-sphere in 4-space can be identified with a subset of the Grassmann manifold of oriented 2-planes in 4-space by associating each great circle fiber with the 2-plane it lies in. This Grassmann manifold can be identified with the space $S^2 \times S^2$. H. Gluck and F. Warner, in *Great circle fibrations of the three sphere*, Duke Math. J. **50** (1983), 107–132, have shown that the subsets of this Grassmann manifold which correspond to great circle fibrations can be interpreted as the graphs of distance decreasing maps from S^2 and S^2 and that Hopf fibrations correspond to constant maps.

This note characterizes explicitly the maps which correspond to "skew-Hopf" fibrations: those fibrations of the 3-sphere obtained from Hopf fibrations by applying a linear transformation of 4-space followed by projection of the fibers back to the unit 3-sphere.

1. Each great circle fibration of the unit 3-sphere in $\mathbf{R}^4$ can be identified with a subset of the Grassmann manifold $G_{2,4}$ of oriented 2-planes in 4-space by associating each great circle fiber with the 2-plane it lies in. It is well known that $G_{2,4}$ can be identified with $S^2 \times S^2$; Gluck and Warner [**G-W**] have shown that the subsets of $G_{2,4}$ which correspond to great circle fibrations can be interpreted as the graphs of distance decreasing maps from S^2 to S^2. In particular, the Hopf fibrations correspond to constant maps.

In this note we describe explicitly all subsets of $G_{2,4}$ which correspond to "skew-Hopf" fibrations; those fibrations of S^3 obtained from Hopf fibrations by applying a linear transformation of $\mathbf{R}^4$ followed by projection of the fibers back to the unit sphere (see **G-W**]). We use the natural embedding of $G_{2,4}$ into $\Lambda^2\mathbf{R}^4$ to show that all such subsets are the intersections of 4-dimensional linear subspaces of $\Lambda^2\mathbf{R}^4$ with $G_{2,4}$. We define a symmetric, nondegenerate, nondefinite form Q and show that a 4-dimensional linear subspace intersects $G_{2,4}$ in a "skew-Hopf subset" if and only if Q restricted to the subspace has signature $(+,-,-,-)$ or $(+,+,+,-)$. The proof consists of showing that the Hopf fibration is the intersection of a specific linear 4-dimensional subspace of $\Lambda^2\mathbf{R}^4$ and determining the orbit of this subspace under the action induced on $\Lambda^2\mathbf{R}^4$ by the affine linear group of $\mathbf{R}^4$. In addition, we obtain a geometric picture of the distance decreasing maps from S^2 to S^2 which correspond to skew-Hopf fibrations. It follows easily from this picture that such maps have convex images.

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2. Notation and preliminaries. If $e_1, \ldots, e_4$ is an orthonormal basis for $\mathbf{R}^4$, then $\{e_i \wedge e_j | 1 \leqslant i < j \leqslant 4\}$ forms an orthonormal basis for the space of bivectors $\Lambda^2\mathbf{R}^4$ and specifies an isomorphism of $\Lambda^2\mathbf{R}^4$ with $\mathbf{R}^6$. In addition to the usual inner product induced on $\Lambda^2\mathbf{R}^4$, there is a second bilinear form Q obtained from the map $\Lambda^2\mathbf{R}^4 \times \Lambda^2\mathbf{R}^4 \to \Lambda^4\mathbf{R}^4 \to \mathbf{R}$ which takes the wedge product of two bivectors of $\mathbf{R}^4$ to a scalar times the volume form of $\mathbf{R}^4$. This bilinear form is symmetric because the wedge product of bivectors is commutative.

It is easily checked that the Plücker basis

$$i_1 = \frac{e_1 \wedge e_2 + e_3 \wedge e_4}{\sqrt{2}}, \qquad i_4 = \frac{e_1 \wedge e_2 - e_3 \wedge e_4}{\sqrt{2}},$$
$$i_2 = \frac{e_1 \wedge e_3 - e_2 \wedge e_4}{\sqrt{2}}, \qquad i_5 = \frac{e_1 \wedge e_3 + e_2 \wedge e_4}{\sqrt{2}},$$
$$i_3 = \frac{e_1 \wedge e_4 + e_2 \wedge e_3}{\sqrt{2}}, \qquad i_6 = \frac{e_1 \wedge e_4 - e_2 \wedge e_3}{\sqrt{2}},$$

is an orthonormal basis which diagonalizes the matrix representing Q. On the subspace $\Lambda^2_+\mathbf{R}^4$ spanned by i_1, i_2 and i_3, Q has eigenvalue $+1$; on the subspace $\Lambda^2_-\mathbf{R}^4$ spanned by i_4, i_5 and i_6, the eigenvalue is -1.

Further calculation shows that a bivector ω in $\Lambda^2\mathbf{R}^4$ can be written as the wedge product of two vectors in $\mathbf{R}^4$ if and only if $Q(\omega, \omega) = 0$; these are the *simple* bivectors of $\Lambda^2\mathbf{R}^4$. Identifying the oriented 2-planes of $\mathbf{R}^4$ with the wedge product of an orthonormal basis which spans it (and possesses the same orientation) yields a one-to-one correspondence between $G_{2,4}$ and the simple bivectors of $\Lambda^2\mathbf{R}^4$ which have unit length. The simple bivectors of unit length form the set

$$S^2_+(1/\sqrt{2}) \times S^2_-(1/\sqrt{2}) \subseteq \Lambda^2_+\mathbf{R}^4 \oplus \Lambda^2_-\mathbf{R}^4.$$

It is convenient to represent elements of $\Lambda^2\mathbf{R}^4$ as a pair of vectors in $\mathbf{R}^3$; that is, $\omega \equiv (\omega^+, \omega^-)$, where ω^+ is the projection of ω onto $\Lambda^2_+\mathbf{R}^4$ and ω^- is the projection onto $\Lambda^2_-\mathbf{R}^4$. In this notation $Q(\omega, \eta) = \langle \omega^+, \eta^+ \rangle - \langle \omega^-, \eta^- \rangle$ and $\langle \omega, \eta \rangle = \langle \omega^+, \eta^+ \rangle + \langle \omega^-, \eta^- \rangle$. The star operator (*) is an involution defined by $*\omega = *(\omega^+, \omega^-) = (\omega^+, -\omega^-)$ which satisfies $Q(\omega, \eta) = \langle \omega, *\eta \rangle$.

It will also be useful to understand the action of elements of the special orthogonal groups $\mathrm{SO}(\mathbf{R}^4)$ and the special linear groups $\mathrm{SL}(\mathbf{R}^4)$ on $\Lambda^2\mathbf{R}^4$. Since elements of $\mathrm{SO}(\mathbf{R}^4)$ preserve both the inner product on $\Lambda^2\mathbf{R}^4$ and the bilinear form Q, one can show by direct calculation that the action of $\mathrm{SO}(\mathbf{R}^4)$ on $\Lambda^2\mathbf{R}^4$ is to rotate each eigenspace of Q independently and that the map $\mathrm{SO}(\mathbf{R}^4) \to \mathrm{SO}(\mathbf{R}^3) \times \mathrm{SO}(\mathbf{R}^3)$ is a double covering. For example, rotation in the e_1, e_2 and e_3, e_4 planes simultaneously by an angle θ induces a rotation in the i_2, i_3 plane of $\Lambda^2\mathbf{R}^4$ through an angle of 2θ and leaves the other basis vectors fixed.

Elements of $\mathrm{SL}(\mathbf{R}^4)$ preserve the bilinear form Q, but not necessarily the inner product. The element of $\mathrm{SL}(\mathbf{R}^4)$ which multiplies the e_1, e_2 plane by e^{ϕ} and the e_3, e_4 plane by $e^{-\phi}$ induces an action on $\Lambda^2\mathbf{R}^4$ which sends i_1 to $(\cosh 2\phi)i_1 + (\sinh 2\phi)i_4$ and i_4 to $(\sinh 2\phi)i_1 + (\cosh 2\phi)i_4$, while leaving the other vectors fixed. These facts can be verified by direct calculation and will be used in the proof of Lemma 2.

LEMMA 1. *The Hopf fibration corresponds to the subset*

$$\left\{\left(i_1/\sqrt{2}\,, v\right) \in S^2_+(1/\sqrt{2}) \times S^2_-(1/\sqrt{2})\,\middle|\, v \in S^2_-\right\} \subseteq G_{2,4}.$$

This subset is topologically $\{\text{point}\} \times S^2$ *and can be considered as the graph of the constant map from* S^2_- *to* S^2_+. *It is one component of the intersection with* $G_{2,4}$ *of the* 4-*dimensional subspace spanned by* i_1, i_4, i_5, i_6. *The other component represents the same fibration with the orientation of the fibers reversed.*

PROOF. We identify $\mathbf{R}^4$ with $\mathbf{C}^2$. The unit sphere is the set $\{(z_1, z_2) \in \mathbf{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ and the great circle fibers of the Hopf fibration are the orbits $\{(z_1 e^{i\theta}, z_2 e^{i\theta}) \,|\, \theta \in [0, 2\pi)\}$. Multiplication of (z_1, z_2) by $e^{i\theta}$ in this way is equivalent to rotating in the e_1, e_2 and e_3, e_4 planes simultaneously by an angle θ. All 2-planes left fixed by this action correspond to fibers of the Hopf fibration and to those points in $G_{2,4}$ which remain fixed under the induced action. This set of fixed points is the intersection of $G_{2,4}$ with the linear subspace spanned by i_1, i_4, i_5, and i_6 as can be found by direct calculation. Topologically it is $S^0 \times S^2$. Each component corresponds to a different orientation on the great circle fibers. (See **[G-W]** for another derivation of this.) When restricted to this subspace Q has signature $(+, -, -, -)$.

LEMMA 2. *The action of an element* $g \in \mathrm{SL}(\mathbf{R}^4)$ *induces a linear action* $\hat{g}$ *on* $\Lambda^2\mathbf{R}^4$ *which preserves the form* Q. *Thus* $\hat{g}$, *acting on one of the* 4-*dimensional linear subspaces* L_4 *of* $\Lambda^2\mathbf{R}^4$, *yields a subspace*, $\hat{g}(L_4)$ *such that* $Q|_{L_4}$ *and* $Q|_{\hat{g}(L_4)}$ *have the same signature*. *Conversely, if* $Q|_{L_4}$ *and* $Q|_{L'_4}$ *have the same signature, then there is an element* g *of* $\mathrm{SL}(\mathbf{R}^4)$ *which induces an action* $\hat{g}$ *which takes* L_4 *to* L'_4.

PROOF. For simple bivectors $\omega = v_1 \wedge v_2$ and $\eta = v_3 \wedge v_4$ and $g \in \mathrm{SL}(\mathbf{R}^4)$, we have, since $\det g = 1$,

$$\begin{aligned} Q(\hat{g}(\omega), \hat{g}(\eta)) e_1 \wedge e_2 \wedge e_3 \wedge e_4 &= g(v_1) \wedge g(v_2) \wedge g(v_3) \wedge g(v_4) \\ &= (\det g) v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &= Q(\omega, \eta) e_1 \wedge e_2 \wedge e_3 \wedge e_4. \end{aligned}$$

The linearity properties of Q and $\hat{g}$ insure that this holds for linear combinations of simple bivectors also. For the second statement one can show that the action of $\mathrm{SL}(\mathbf{R}^4)$ on $G_{2,4}$ is one component of the automorphism group of Q and, hence, by Witt's theorem **[A**, p. 121**]** there is an element of $\mathrm{SL}(\mathbf{R}^4)$ which takes any L_4 of $G_{2,4}$ to any other L'_4 with the same signature.

To see this directly, without Witt's theorem, we construct adapted frames for L_4 and L'_4 and construct the element of $\mathrm{SL}(\mathbf{R}^4)$ which takes one frame to the other:

For any 4-dimensional linear subspace L_4 we can find elements η_1 and η_2 of $\Lambda^2\mathbf{R}^4$ which satisfy

(1) $\langle \eta_i, \eta_j \rangle = \delta_{ij}$, $i, j = 1, 2$,
(2) $Q(\eta_1, \eta_2) = 0$,
(3) $Q(\eta_i, \lambda) = 0$ for all $\lambda \in L_4$ and $i = 1, 2$.

The 2-dimensional subspace spanned by η_1, η_2 is *Q-orthogonal* to L_4 by (3). These subspaces can be classified by the signature of the restriction of Q to the subspaces. Furthermore, since L_4 and the 2-dimensional subspace are Q orthogonal, one can extend η_1, η_2 to a basis of $\Lambda^2\mathbf{R}^4$ which diagonalizes Q, and therefore the signature of Q on the 2-dimensional space determines the signature of Q on the 4-dimensional space and vice versa.

The construction of η_1 and η_2 satisfying (1)–(3) proceeds as follows: Choose orthonormal vectors $\omega_1, \omega_2 \in \Lambda^2\mathbf{R}^4$ perpendicular to L_4 (i.e., $\langle \omega_1, \lambda\rangle = \langle \omega_2, \lambda\rangle = 0$ for all $\lambda \in L_4$). By a rotation in the ω_1, ω_2 plane, we obtain a new basis $\tilde{\omega}_1, \tilde{\omega}_2$ which diagonalizes Q (i.e., $Q(\tilde{\omega}_1, \tilde{\omega}_2) = 0$). Let $\eta_1 = *\tilde{\omega}_1$ and $\eta_2 = *\tilde{\omega}_2$ and use the fact that $\langle *\omega, *\eta\rangle = \langle \omega, \eta\rangle$ and $Q(\omega, \eta) = \langle \omega, *\eta\rangle$ to verify (1)–(3).

We now construct an action of $\mathrm{SL}(\mathbf{R}^4)$ which moves the vector η_1, η_2 to a multiple of i_1, i_2 provided $Q(\eta_1, \eta_1) > 0$ and $Q(\eta_2, \eta_2) > 0$.

We represent η_1 as (η_1^+, η_1^-) and η_2 as (η_2^+, η_2^-). There is an element of g of $\mathrm{SO}(\mathbf{R}^4)$ which rotates $\Lambda^2\mathbf{R}^4$ so that η_1^+ and η_2^+ are parallel with i_1 and i_2, respectively, and which simultaneously rotates $\Lambda^2_-\mathbf{R}^4$ so that η_1^- and η_2^- are aligned with i_4 and i_5. (Note that $\langle \eta_1^+, \eta_2^+\rangle = \langle \eta_1^-, \eta_2^-\rangle = 0$ as a consequence of (1) and (3).) Direct computation shows that the diagonal matrix with entries $(e^{\phi}, e^{\phi}, e^{-\phi}, e^{-\phi})$ represents an element of $\mathrm{SL}(\mathbf{R}^4)$ which takes $g(\eta_1) = ai_1 + bi_4$ to $(a\cosh 2\phi + b\sinh 2\phi)i_1 + (a\sinh 2\phi + b\cosh 2\phi)i_4$ and leaves $g(\eta_2)$ fixed (see §2). Since $Q(\eta_1, \eta_1) = Q(g(\eta_1), g(\eta_1)) = a^2 - b^2 > 0$, it is possible to find a ϕ which makes the coefficient of i_4 zero. Similarly, we can find an element of $\mathrm{SL}(\mathbf{R}^4)$ which takes $g(\eta_2)$ to a multiple of i_2. The composition of these three transformations is an element g of $\mathrm{SL}(\mathbf{R}^4)$ whose action on $\Lambda^2\mathbf{R}^4$ takes η_1 to a multiple of i_1 and η_2 to a multiple of i_2.

Since this transformation preserves Q it takes L_4 to the space spanned by i_3, i_4, i_5, and i_6.

For any other L_4' with the same signature there is an action g' of $\mathrm{SL}(\mathbf{R}^4)$ which also takes L_4' to the space spanned by i_3, i_4, i_5 and i_6 and, therefore, $(g')^{-1} \circ g$ takes L_4 to L_4'. This completes the proof when $Q|_{L_4}$ has signature $(+, -, -, -)$; similar arguments handle the other possible signatures of $Q|_{L_4}$.

From Lemmas 1 and 2 we immediately deduce:

PROPOSITION 3. *There is a one-to-one correspondence between skew-Hopf fibrations and those* 4*-dimensional subspaces* L_4 *of* $\Lambda^2\mathbf{R}^4$ *such that* $Q|_{L_4}$ *has signature* $(+, +, +, -)$ *or* $(+, -, -, -)$.

PROOF. The element $g \in \mathrm{SL}(\mathbf{R}^4)$ which takes the Hopf fibration in Lemma 1 to a skew Hopf fibration also takes the 4-plane spanned by i_1, i_4, i_5, i_6 to a 4-dimensional subspace whose intersection with $G_{2,4}$ corresponds to the skew-Hopf fibration. When Q is restricted to this subspace it has signature $(+, -, -, -)$. On the other hand, given any 4-dimensional subspace with this signature, there is a g in $\mathrm{SL}(\mathbf{R}^4)$ such that $\hat{g}$ takes $\{i_1, i_4, i_5, i_6\}$ to the given subspace (Lemma 2) and g takes the standard Hopf fibration to a corresponding skew Hopf fibration.

The reflection which takes e_4 to $-e_4$ takes the standard Hopf fibration to one with reverse screw sense and takes the corresponding 4-plane to one spanned by i_1, i_2, i_3, i_4 which has signature $(+, +, +, -)$. This completes the proof.

We know from [G-W] that all great circle fibrations of S^3 correspond to the graph of distance decreasing maps from S^2 to S^2. The next task is to describe explicitly the maps corresponding to skew-Hopf fibrations. Choosing the appropriate coordinates in $\Lambda^2\mathbf{R}^4$ makes this easy.

Assume that L_4 is a fixed 4-dimensional subspace which corresponds to a skew-Hopf fibration and for which $Q|_{L_4}$ has signature $(+,-,-,-)$. Choose the unit vectors η_1 and η_2 of $\Lambda^2\mathbf{R}^4$ which span the space Q-orthogonal to L_4 and which satisfy (1)–(3). In terms of the decomposition $\Lambda^2\mathbf{R}^4 = \Lambda^2_+\mathbf{R}^4 \oplus \Lambda^2_-\mathbf{R}^4 = \mathbf{R}^3 \oplus \mathbf{R}^3$, we write $\eta_1 = (\eta_1^+, \eta_1^-)$ and $\eta_2 = (\eta_2^+, \eta_2^-)$. From (1) and (2) we have

$$\langle \eta_1, \eta_2 \rangle = \langle \eta_1^+, \eta_2^+ \rangle + \langle \eta_1^-, \eta_2^- \rangle = 0$$

and

$$Q(\eta_1, \eta_2) = \langle \eta_1^+, \eta_2^+ \rangle - \langle \eta_1^-, \eta_2^- \rangle = 0,$$

and therefore η_1^+, η_2^+, η_1^- and η_2^- are orthogonal (but not necessarily of unit length). We can choose a coordinate system $(x_1, x_2, x_3, y_1, y_2, y_3)$ of $\Lambda^2_+\mathbf{R}^4 \oplus \Lambda^2_-\mathbf{R}^4$ so that η_i^+ is parallel to the x_i-axis and η_i^- is parallel to the y_i-axis for $i = 1, 2$. η_2^+ would be represented in coordinates by $(0, |\eta_2^+|, 0, 0, 0, 0)$, where $|\eta_2^+|$ is the length of η_2^+.

In terms of these coordinates we can express an arbitrary element $\omega = (x_1, x_2, x_3, y_1, y_2, y_3)$ and can write the following conditions in coordinate terms:

If ω is in $G_{2,4}$ then

(5) $$0 = Q(\omega, \omega) = x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2$$

and

(6) $$1 = \langle \omega, \omega \rangle = x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2.$$

If ω is also in L_4 and Q-orthogonal to η_1 and η_2, then

(7) $$0 = Q(\omega, \eta_i) = x_i|\eta_i^+| - y_i|\eta_i^-|, \qquad i = 1, 2.$$

The condition that $Q(\eta_i, \eta_i) > 0$ becomes

(8) $$0 < Q(\eta_i, \eta_i) = |\eta_i^+|^2 - |\eta_i^-|^2, \quad i = 1, 2.$$

We rewrite (7) as

(9) $$x_1 = \left(|\eta_1^-|/|\eta_1^+|\right) y_1$$

and

(10) $$x_2 = \left(|\eta_2^-|/|\eta_2^+|\right) y_2,$$

and from (5) and (6) we also have

(11) $$x_3 = +\sqrt{1/2 - x_1^2 - x_2^2}\,.$$

Thus each element ω in the plane L_4 is in the graph of the function defined from $S^2(1/\sqrt{2})$ to $S^2(1/\sqrt{2})$ by (9)–(11), or is in the graph of the function defined by changing the sign in front of the radical sign in (11). We can describe these equations geometrically as follows:

PROPOSITION 4. *Each skew-Hopf fibration corresponds to a distance decreasing map from* $S^2(1/\sqrt{2}) \to S^2(1/\sqrt{2})$ *which can be decomposed as*: (a) *an orthogonal projection P to a plane through the center of the sphere*; *followed by* (b) *a distance decreasing linear map A from one 2-plane to another*; *and finally* (c) *inverse projection onto* $S^2(1/\sqrt{2})$.

PROOF. The orthogonal projection in the coordinate system above is onto the (y_1, y_2)-plane. Equations (9) and (10) define the linear map and are distance decreasing because $|\eta_i^-|/|\eta_i^+| < 1$ from (8). The choice of inverse projection P^{-1} determines the sign in (11).

PROPOSITION 5. *The maps* $S^2 \to S^2$ *which correspond to skew-Hopf fibrations have a convex image.*

PROOF. All such maps can be written in the form $P^{-1} \circ A \circ P$ according to Proposition 4. The image of $A \circ P$ is the interior of an ellipse E which lies inside the unit disk. It remains to show that P^{-1} maps an ellipse to a convex region in the sphere. Let x and y be two points in the image of $P^{-1} \circ A \circ P$ and let γ represent the great circle passing through these points. Considering the projection onto the plane we see that $P(\gamma)$ is an ellipse inscribed in the unit circle which intersects the ellipse E in at most four points.

The portion of γ which lies in the northern hemisphere projects to a half-ellipse whose endpoints lie on the unit circle and which, by symmetry, intersects the boundary of E only twice. The segment from $P(x)$ to $P(y)$ must lie in the interior of E. It follows that the distance minimizing geodesic from x to y lies inside $\text{Im}(P^{-1} \circ A \circ P)$ and the proposition is proved.

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