

A NOTE ON SKEW-HOPF FIBRATIONS

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ABSTRACT. Each great circle fibration of the unit 3-sphere in 4-space can be identified with a subset of the Grassmann manifold of oriented 2-planes in 4-space by associating each great circle fiber with the 2-plane it lies in. This Grassmann manifold can be identified with the space $S^2 \times S^2$. H. Gluck and F. Warner, in *Great circle fibrations of the three sphere*, Duke Math. J. **50** (1983), 107–132, have shown that the subsets of this Grassmann manifold which correspond to great circle fibrations can be interpreted as the graphs of distance decreasing maps from S^2 and S^2 and that Hopf fibrations correspond to constant maps.

This note characterizes explicitly the maps which correspond to “skew-Hopf” fibrations: those fibrations of the 3-sphere obtained from Hopf fibrations by applying a linear transformation of 4-space followed by projection of the fibers back to the unit 3-sphere.

1. Each great circle fibration of the unit 3-sphere in \mathbf{R}^4 can be identified with a subset of the Grassmann manifold $G_{2,4}$ of oriented 2-planes in 4-space by associating each great circle fiber with the 2-plane it lies in. It is well known that $G_{2,4}$ can be identified with $S^2 \times S^2$; Gluck and Warner [G-W] have shown that the subsets of $G_{2,4}$ which correspond to great circle fibrations can be interpreted as the graphs of distance decreasing maps from S^2 to S^2 . In particular, the Hopf fibrations correspond to constant maps.

In this note we describe explicitly all subsets of $G_{2,4}$ which correspond to “skew-Hopf” fibrations; those fibrations of S^3 obtained from Hopf fibrations by applying a linear transformation of \mathbf{R}^4 followed by projection of the fibers back to the unit sphere (see G-W). We use the natural embedding of $G_{2,4}$ into $\Lambda^2 \mathbf{R}^4$ to show that all such subsets are the intersections of 4-dimensional linear subspaces of $\Lambda^2 \mathbf{R}^4$ with $G_{2,4}$. We define a symmetric, nondegenerate, nondefinite form Q and show that a 4-dimensional linear subspace intersects $G_{2,4}$ in a “skew-Hopf subset” if and only if Q restricted to the subspace has signature $(+, -, -, -)$ or $(+, +, +, -)$. The proof consists of showing that the Hopf fibration is the intersection of a specific linear 4-dimensional subspace of $\Lambda^2 \mathbf{R}^4$ and determining the orbit of this subspace under the action induced on $\Lambda^2 \mathbf{R}^4$ by the affine linear group of \mathbf{R}^4 . In addition, we obtain a geometric picture of the distance decreasing maps from S^2 to S^2 which correspond to skew-Hopf fibrations. It follows easily from this picture that such maps have convex images.

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2. Notation and preliminaries. If e_1, \dots, e_4 is an orthonormal basis for \mathbf{R}^4 , then $\{e_i \wedge e_j | 1 \leq i < j \leq 4\}$ forms an orthonormal basis for the space of bivectors $\Lambda^2 \mathbf{R}^4$ and specifies an isomorphism of $\Lambda^2 \mathbf{R}^4$ with \mathbf{R}^6 . In addition to the usual inner product induced on $\Lambda^2 \mathbf{R}^4$, there is a second bilinear form Q obtained from the map $\Lambda^2 \mathbf{R}^4 \times \Lambda^2 \mathbf{R}^4 \rightarrow \Lambda^4 \mathbf{R}^4 \rightarrow \mathbf{R}$ which takes the wedge product of two bivectors of \mathbf{R}^4 to a scalar times the volume form of \mathbf{R}^4 . This bilinear form is symmetric because the wedge product of bivectors is commutative.

It is easily checked that the Plücker basis

$$\begin{aligned} i_1 &= \frac{e_1 \wedge e_2 + e_3 \wedge e_4}{\sqrt{2}}, & i_4 &= \frac{e_1 \wedge e_2 - e_3 \wedge e_4}{\sqrt{2}}, \\ i_2 &= \frac{e_1 \wedge e_3 - e_2 \wedge e_4}{\sqrt{2}}, & i_5 &= \frac{e_1 \wedge e_3 + e_2 \wedge e_4}{\sqrt{2}}, \\ i_3 &= \frac{e_1 \wedge e_4 + e_2 \wedge e_3}{\sqrt{2}}, & i_6 &= \frac{e_1 \wedge e_4 - e_2 \wedge e_3}{\sqrt{2}}, \end{aligned}$$

is an orthonormal basis which diagonalizes the matrix representing Q . On the subspace $\Lambda_+^2 \mathbf{R}^4$ spanned by i_1, i_2 and i_3 , Q has eigenvalue $+1$; on the subspace $\Lambda_-^2 \mathbf{R}^4$ spanned by i_4, i_5 and i_6 , the eigenvalue is -1 .

Further calculation shows that a bivector ω in $\Lambda^2 \mathbf{R}^4$ can be written as the wedge product of two vectors in \mathbf{R}^4 if and only if $Q(\omega, \omega) = 0$; these are the *simple* bivectors of $\Lambda^2 \mathbf{R}^4$. Identifying the oriented 2-planes of \mathbf{R}^4 with the wedge product of an orthonormal basis which spans it (and possesses the same orientation) yields a one-to-one correspondence between $G_{2,4}$ and the simple bivectors of $\Lambda^2 \mathbf{R}^4$ which have unit length. The simple bivectors of unit length form the set

$$S_+^2(1/\sqrt{2}) \times S_-^2(1/\sqrt{2}) \subseteq \Lambda_+^2 \mathbf{R}^4 \oplus \Lambda_-^2 \mathbf{R}^4.$$

It is convenient to represent elements of $\Lambda^2 \mathbf{R}^4$ as a pair of vectors in \mathbf{R}^3 ; that is, $\omega \equiv (\omega^+, \omega^-)$, where ω^+ is the projection of ω onto $\Lambda_+^2 \mathbf{R}^4$ and ω^- is the projection onto $\Lambda_-^2 \mathbf{R}^4$. In this notation $Q(\omega, \eta) = \langle \omega^+, \eta^+ \rangle - \langle \omega^-, \eta^- \rangle$ and $\langle \omega, \eta \rangle = \langle \omega^+, \eta^+ \rangle + \langle \omega^-, \eta^- \rangle$. The star operator ($*$) is an involution defined by $*\omega = *(\omega^+, \omega^-) = (\omega^+, -\omega^-)$ which satisfies $Q(\omega, \eta) = \langle \omega, *\eta \rangle$.

It will also be useful to understand the action of elements of the special orthogonal groups $SO(\mathbf{R}^4)$ and the special linear groups $SL(\mathbf{R}^4)$ on $\Lambda^2 \mathbf{R}^4$. Since elements of $SO(\mathbf{R}^4)$ preserve both the inner product on $\Lambda^2 \mathbf{R}^4$ and the bilinear form Q , one can show by direct calculation that the action of $SO(\mathbf{R}^4)$ on $\Lambda^2 \mathbf{R}^4$ is to rotate each eigenspace of Q independently and that the map $SO(\mathbf{R}^4) \rightarrow SO(\mathbf{R}^3) \times SO(\mathbf{R}^3)$ is a double covering. For example, rotation in the e_1, e_2 and e_3, e_4 planes simultaneously by an angle θ induces a rotation in the i_2, i_3 plane of $\Lambda^2 \mathbf{R}^4$ through an angle of 2θ and leaves the other basis vectors fixed.

Elements of $SL(\mathbf{R}^4)$ preserve the bilinear form Q , but not necessarily the inner product. The element of $SL(\mathbf{R}^4)$ which multiplies the e_1, e_2 plane by e^ϕ and the e_3, e_4 plane by $e^{-\phi}$ induces an action on $\Lambda^2 \mathbf{R}^4$ which sends i_1 to $(\cosh 2\phi)i_1 + (\sinh 2\phi)i_4$ and i_4 to $(\sinh 2\phi)i_1 + (\cosh 2\phi)i_4$, while leaving the other vectors fixed. These facts can be verified by direct calculation and will be used in the proof of Lemma 2.

LEMMA 1. *The Hopf fibration corresponds to the subset*

$$\left\{ (i_1/\sqrt{2}, v) \in S_+^2(1/\sqrt{2}) \times S_-^2(1/\sqrt{2}) \mid v \in S_-^2 \right\} \subseteq G_{2,4}.$$

This subset is topologically $\{\text{point}\} \times S^2$ and can be considered as the graph of the constant map from S_-^2 to S_+^2 . It is one component of the intersection with $G_{2,4}$ of the 4-dimensional subspace spanned by i_1, i_4, i_5, i_6 . The other component represents the same fibration with the orientation of the fibers reversed.

PROOF. We identify \mathbf{R}^4 with \mathbf{C}^2 . The unit sphere is the set $\{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ and the great circle fibers of the Hopf fibration are the orbits $\{(z_1 e^{i\theta}, z_2 e^{i\theta}) \mid \theta \in [0, 2\pi)\}$. Multiplication of (z_1, z_2) by $e^{i\theta}$ in this way is equivalent to rotating in the e_1, e_2 and e_3, e_4 planes simultaneously by an angle θ . All 2-planes left fixed by this action correspond to fibers of the Hopf fibration and to those points in $G_{2,4}$ which remain fixed under the induced action. This set of fixed points is the intersection of $G_{2,4}$ with the linear subspace spanned by i_1, i_4, i_5 , and i_6 as can be found by direct calculation. Topologically it is $S^0 \times S^2$. Each component corresponds to a different orientation on the great circle fibers. (See [G-W] for another derivation of this.) When restricted to this subspace Q has signature $(+, -, -, -)$.

LEMMA 2. *The action of an element $g \in \text{SL}(\mathbf{R}^4)$ induces a linear action \hat{g} on $\Lambda^2 \mathbf{R}^4$ which preserves the form Q . Thus \hat{g} , acting on one of the 4-dimensional linear subspaces L_4 of $\Lambda^2 \mathbf{R}^4$, yields a subspace, $\hat{g}(L_4)$ such that $Q|_{L_4}$ and $Q|_{\hat{g}(L_4)}$ have the same signature. Conversely, if $Q|_{L_4}$ and $Q|_{L'_4}$ have the same signature, then there is an element g of $\text{SL}(\mathbf{R}^4)$ which induces an action \hat{g} which takes L_4 to L'_4 .*

PROOF. For simple bivectors $\omega = v_1 \wedge v_2$ and $\eta = v_3 \wedge v_4$ and $g \in \text{SL}(\mathbf{R}^4)$, we have, since $\det g = 1$,

$$\begin{aligned} Q(\hat{g}(\omega), \hat{g}(\eta)) e_1 \wedge e_2 \wedge e_3 \wedge e_4 &= g(v_1) \wedge g(v_2) \wedge g(v_3) \wedge g(v_4) \\ &= (\det g) v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &= Q(\omega, \eta) e_1 \wedge e_2 \wedge e_3 \wedge e_4. \end{aligned}$$

The linearity properties of Q and \hat{g} insure that this holds for linear combinations of simple bivectors also. For the second statement one can show that the action of $\text{SL}(\mathbf{R}^4)$ on $G_{2,4}$ is one component of the automorphism group of Q and, hence, by Witt's theorem [A, p. 121] there is an element of $\text{SL}(\mathbf{R}^4)$ which takes any L_4 of $G_{2,4}$ to any other L'_4 with the same signature.

To see this directly, without Witt's theorem, we construct adapted frames for L_4 and L'_4 and construct the element of $\text{SL}(\mathbf{R}^4)$ which takes one frame to the other:

For any 4-dimensional linear subspace L_4 we can find elements η_1 and η_2 of $\Lambda^2 \mathbf{R}^4$ which satisfy

- (1) $\langle \eta_i, \eta_j \rangle = \delta_{ij}, i, j = 1, 2,$
- (2) $Q(\eta_1, \eta_2) = 0,$
- (3) $Q(\eta_i, \lambda) = 0$ for all $\lambda \in L_4$ and $i = 1, 2.$

The 2-dimensional subspace spanned by η_1, η_2 is Q -orthogonal to L_4 by (3). These subspaces can be classified by the signature of the restriction of Q to the subspaces. Furthermore, since L_4 and the 2-dimensional subspace are Q orthogonal, one can extend η_1, η_2 to a basis of $\Lambda^2 \mathbf{R}^4$ which diagonalizes Q , and therefore the signature of Q on the 2-dimensional space determines the signature of Q on the 4-dimensional space and vice versa.

The construction of η_1 and η_2 satisfying (1)–(3) proceeds as follows: Choose orthonormal vectors $\omega_1, \omega_2 \in \Lambda^2 \mathbf{R}^4$ perpendicular to L_4 (i.e., $\langle \omega_1, \lambda \rangle = \langle \omega_2, \lambda \rangle = 0$ for all $\lambda \in L_4$). By a rotation in the ω_1, ω_2 plane, we obtain a new basis $\tilde{\omega}_1, \tilde{\omega}_2$ which diagonalizes Q (i.e., $Q(\tilde{\omega}_1, \tilde{\omega}_2) = 0$). Let $\eta_1 = *\tilde{\omega}_1$ and $\eta_2 = *\tilde{\omega}_2$ and use the fact that $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$ and $Q(\omega, \eta) = \langle \omega, *\eta \rangle$ to verify (1)–(3).

We now construct an action of $SL(\mathbf{R}^4)$ which moves the vector η_1, η_2 to a multiple of i_1, i_2 provided $Q(\eta_1, \eta_1) > 0$ and $Q(\eta_2, \eta_2) > 0$.

We represent η_1 as (η_1^+, η_1^-) and η_2 as (η_2^+, η_2^-) . There is an element of g of $SO(\mathbf{R}^4)$ which rotates $\Lambda^2 \mathbf{R}^4$ so that η_1^+ and η_2^+ are parallel with i_1 and i_2 , respectively, and which simultaneously rotates $\Lambda^2 \mathbf{R}^4$ so that η_1^- and η_2^- are aligned with i_4 and i_5 . (Note that $\langle \eta_1^+, \eta_2^+ \rangle = \langle \eta_1^-, \eta_2^- \rangle = 0$ as a consequence of (1) and (3).) Direct computation shows that the diagonal matrix with entries $(e^\phi, e^\phi, e^{-\phi}, e^{-\phi})$ represents an element of $SL(\mathbf{R}^4)$ which takes $g(\eta_1) = ai_1 + bi_4$ to $(a \cosh 2\phi + b \sinh 2\phi)i_1 + (a \sinh 2\phi + b \cosh 2\phi)i_4$ and leaves $g(\eta_2)$ fixed (see §2). Since $Q(\eta_1, \eta_1) = Q(g(\eta_1), g(\eta_1)) = a^2 - b^2 > 0$, it is possible to find a ϕ which makes the coefficient of i_4 zero. Similarly, we can find an element of $SL(\mathbf{R}^4)$ which takes $g(\eta_2)$ to a multiple of i_2 . The composition of these three transformations is an element g of $SL(\mathbf{R}^4)$ whose action on $\Lambda^2 \mathbf{R}^4$ takes η_1 to a multiple of i_1 and η_2 to a multiple of i_2 .

Since this transformation preserves Q it takes L_4 to the space spanned by i_3, i_4, i_5 , and i_6 .

For any other L'_4 with the same signature there is an action g' of $SL(\mathbf{R}^4)$ which also takes L'_4 to the space spanned by i_3, i_4, i_5 and i_6 and, therefore, $(g')^{-1} \circ g$ takes L_4 to L'_4 . This completes the proof when $Q|_{L_4}$ has signature $(+, -, -, -)$; similar arguments handle the other possible signatures of $Q|_{L_4}$.

From Lemmas 1 and 2 we immediately deduce:

PROPOSITION 3. *There is a one-to-one correspondence between skew-Hopf fibrations and those 4-dimensional subspaces L_4 of $\Lambda^2 \mathbf{R}^4$ such that $Q|_{L_4}$ has signature $(+, +, +, -)$ or $(+, -, -, -)$.*

PROOF. The element $g \in SL(\mathbf{R}^4)$ which takes the Hopf fibration in Lemma 1 to a skew Hopf fibration also takes the 4-plane spanned by i_1, i_4, i_5, i_6 to a 4-dimensional subspace whose intersection with $G_{2,4}$ corresponds to the skew-Hopf fibration. When Q is restricted to this subspace it has signature $(+, -, -, -)$. On the other hand, given any 4-dimensional subspace with this signature, there is a g in $SL(\mathbf{R}^4)$ such that \hat{g} takes $\{i_1, i_4, i_5, i_6\}$ to the given subspace (Lemma 2) and g takes the standard Hopf fibration to a corresponding skew Hopf fibration.

The reflection which takes e_4 to $-e_4$ takes the standard Hopf fibration to one with reverse screw sense and takes the corresponding 4-plane to one spanned by i_1, i_2, i_3, i_4 which has signature $(+, +, +, -)$. This completes the proof.

We know from [G-W] that all great circle fibrations of S^3 correspond to the graph of distance decreasing maps from S^2 to S^2 . The next task is to describe explicitly the maps corresponding to skew-Hopf fibrations. Choosing the appropriate coordinates in $\Lambda^2 \mathbf{R}^4$ makes this easy.

Assume that L_4 is a fixed 4-dimensional subspace which corresponds to a skew-Hopf fibration and for which $Q|_{L_4}$ has signature $(+, -, -, -)$. Choose the unit vectors η_1 and η_2 of $\Lambda^2 \mathbf{R}^4$ which span the space Q -orthogonal to L_4 and which satisfy (1)–(3). In terms of the decomposition $\Lambda^2 \mathbf{R}^4 = \Lambda^2_+ \mathbf{R}^4 \oplus \Lambda^2_- \mathbf{R}^4 = \mathbf{R}^3 \oplus \mathbf{R}^3$, we write $\eta_1 = (\eta_1^+, \eta_1^-)$ and $\eta_2 = (\eta_2^+, \eta_2^-)$. From (1) and (2) we have

$$\langle \eta_1, \eta_2 \rangle = \langle \eta_1^+, \eta_2^+ \rangle + \langle \eta_1^-, \eta_2^- \rangle = 0$$

and

$$Q(\eta_1, \eta_2) = \langle \eta_1^+, \eta_2^+ \rangle - \langle \eta_1^-, \eta_2^- \rangle = 0,$$

and therefore $\eta_1^+, \eta_2^+, \eta_1^-$ and η_2^- are orthogonal (but not necessarily of unit length). We can choose a coordinate system $(x_1, x_2, x_3, y_1, y_2, y_3)$ of $\Lambda^2_+ \mathbf{R}^4 \oplus \Lambda^2_- \mathbf{R}^4$ so that η_i^+ is parallel to the x_i -axis and η_i^- is parallel to the y_i -axis for $i = 1, 2$. η_2^+ would be represented in coordinates by $(0, |\eta_2^+|, 0, 0, 0, 0)$, where $|\eta_2^+|$ is the length of η_2^+ .

In terms of these coordinates we can express an arbitrary element $\omega = (x_1, x_2, x_3, y_1, y_2, y_3)$ and can write the following conditions in coordinate terms:

If ω is in $G_{2,4}$ then

$$(5) \quad 0 = Q(\omega, \omega) = x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2$$

and

$$(6) \quad 1 = \langle \omega, \omega \rangle = x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2.$$

If ω is also in L_4 and Q -orthogonal to η_1 and η_2 , then

$$(7) \quad 0 = Q(\omega, \eta_i) = x_i |\eta_i^+| - y_i |\eta_i^-|, \quad i = 1, 2.$$

The condition that $Q(\eta_i, \eta_i) > 0$ becomes

$$(8) \quad 0 < Q(\eta_i, \eta_i) = |\eta_i^+|^2 - |\eta_i^-|^2, \quad i = 1, 2.$$

We rewrite (7) as

$$(9) \quad x_1 = (|\eta_1^-|/|\eta_1^+|) y_1$$

and

$$(10) \quad x_2 = (|\eta_2^-|/|\eta_2^+|) y_2,$$

and from (5) and (6) we also have

$$(11) \quad x_3 = +\sqrt{1/2 - x_1^2 - x_2^2}.$$

Thus each element ω in the plane L_4 is in the graph of the function defined from $S^2(1/\sqrt{2})$ to $S^2(1/\sqrt{2})$ by (9)–(11), or is in the graph of the function defined by changing the sign in front of the radical sign in (11). We can describe these equations geometrically as follows:

PROPOSITION 4. *Each skew-Hopf fibration corresponds to a distance decreasing map from $S^2(1/\sqrt{2}) \rightarrow S^2(1/\sqrt{2})$ which can be decomposed as: (a) an orthogonal projection P to a plane through the center of the sphere; followed by (b) a distance decreasing linear map A from one 2-plane to another; and finally (c) inverse projection onto $S^2(1/\sqrt{2})$.*

PROOF. The orthogonal projection in the coordinate system above is onto the (y_1, y_2) -plane. Equations (9) and (10) define the linear map and are distance decreasing because $|\eta_i^-|/|\eta_i^+| < 1$ from (8). The choice of inverse projection P^{-1} determines the sign in (11).

PROPOSITION 5. *The maps $S^2 \rightarrow S^2$ which correspond to skew-Hopf fibrations have a convex image.*

PROOF. All such maps can be written in the form $P^{-1} \circ A \circ P$ according to Proposition 4. The image of $A \circ P$ is the interior of an ellipse E which lies inside the unit disk. It remains to show that P^{-1} maps an ellipse to a convex region in the sphere. Let x and y be two points in the image of $P^{-1} \circ A \circ P$ and let γ represent the great circle passing through these points. Considering the projection onto the plane we see that $P(\gamma)$ is an ellipse inscribed in the unit circle which intersects the ellipse E in at most four points.

The portion of γ which lies in the northern hemisphere projects to a half-ellipse whose endpoints lie on the unit circle and which, by symmetry, intersects the boundary of E only twice. The segment from $P(x)$ to $P(y)$ must lie in the interior of E . It follows that the distance minimizing geodesic from x to y lies inside $\text{Im}(P^{-1} \circ A \circ P)$ and the proposition is proved.

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