THE TWO MULTIPLICATIONS ON $BU$

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Abstract. We prove that $BU_p$ has only two standard $H$-structures.

The two most familiar $H$-multiplications on $BU$ arise from the direct sum of complex vector bundles and from the tensor product of virtual line bundles; $BU$ with these multiplications will be denoted by $BU^\oplus$ and $BU^\otimes$ respectively [2]. Both multiplications are standard, define a standard $H$-space $(X, e, \mu)$ to be an $H$-space whose rational Pontrjagin ring $H_*(X, \mathbb{Q})$ induced by $\mu$ is both associative and graded commutative. We consider $H$-spaces localized at a fixed prime $p$.

**Theorem 1.1.** Let $m$ be a standard $H$-multiplication on $BU$. Then $BU_p^m$ is $H$-equivalent to $BU_p^\oplus$ or $BU_p^\otimes$.

We recall that two $H$-spaces $(X, e, \mu)$ and $(Y, f, \nu)$ are $H$-equivalent if there exists a homotopy equivalence $k: X \to Y$ such that $\nu(k \times k) = k \mu$. The two classes of multiplications on $BU_p$ in Theorem 1.1 can easily be distinguished for the Frobenius map

$$
\xi: H_2(BU_p^m, \mathbb{Z}/p\mathbb{Z}) \to H_2p(BU_p^m, \mathbb{Z}/p\mathbb{Z})
$$

defined by $\xi(x) = x^p$ is nontrivial when $m = \oplus$ but is the zero homomorphism if $m = \otimes$. Theorem 1.1 completes results about standard multiplications on $BU_p$ proved in [4]. The strongest result in this context was given in §4 of [4]. $X$ is assumed to have the homotopy type of a connected CW-complex with finite skeleta.

**Theorem 1.2.** Let $X$ be a standard $H$-space and $p$ an odd prime. Assume that
(a) $H^*(X, \mathbb{Z}/p\mathbb{Z})$ is a polynomial algebra,
(b) $\dim_{\mathbb{Q}} H_2^j(X, \mathbb{Z}/p\mathbb{Z}) \leq 1$ for all $i$ and the equality holds at least for $1 \leq i \leq p - 1,
(c) \xi: H_2(X, \mathbb{Z}/p\mathbb{Z}) \to H_2p(X, \mathbb{Z}/p\mathbb{Z})$ is nontrivial.

Then $X_p = BU_p^\otimes$ as an $H$-space.

We can now add a companion result for $BU_p^\oplus$.

**Theorem 1.3.** Let $X$ be a standard $H$-space and $p$ an odd prime. Assume that
(a) $H^*(X, \mathbb{Z}/p\mathbb{Z})$ is a polynomial algebra,
(b) $\dim_{\mathbb{Q}} H_2^j(X, \mathbb{Z}/p\mathbb{Z}) \leq 1$ for all $i$ and the equality holds at least for $1 \leq i \leq p$,

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(b) $P_1: QH^{2p}(X, Z/pZ) \to QH^{4p-2}(X, Z/pZ)$ is an isomorphism,

(c) $\xi: H_2(X, Z/pZ) \to H_{2p}(X, Z/pZ)$ is trivial.

Then $X_p = BU_p^\otimes$ as an $H$-space.

It was shown in Theorem 4.4 of [4] that the hypotheses of Theorem 1.3 imply that $X_p$ and $BU_p$ have the same homotopy type. So it remains to show that we can choose a homotopy equivalence which is an $H$-map. At odd primes Theorem 1.2 and Theorem 1.3 imply Theorem 1.1. When $p = 2$ and the Frobenius map is nontrivial the conclusion of Theorem 1.1 follows from the final few lines of [4]. Therefore in order to complete the proofs of both Theorem 1.1 and Theorem 1.3 it remains to show that if $p$ is any prime $X_p = BU_p$ and the Frobenius map

$$\xi: H_2(X, Z/pZ) \to H_{2p}(X, Z/pZ)$$

is trivial, then $X_p = BU_p^\otimes$ as an $H$-space.

The results of this paper can be extended to the classifying spaces of other stable classical groups except that when $p = 2$ we must exclude $BO$ and $BSO$ as the central result used in the proofs is Theorem 3.1 of [4] and this cannot be used when homology 2-torsion is present. Therefore we state results for $BSU$ and $BSp$ only.

**Theorem 1.4.** Let $m$ be a standard multiplication on $BG$, where $G = SU$ or $Sp$. Then $BG_p^m = BG_p^\otimes$ as an $H$-space.

The result corresponding to Theorems 1.2 and 1.3 for $BSp$ is

**Theorem 1.5.** Let $X$ be a standard $H$-space and $p$ an odd prime. Assume that

(a) $H^*(X, Z/pZ)$ is a polynomial algebra,

(b)' $\dim QH^{4i+2}(X, Z/pZ) = 0$ for all $i$,

(b)' $\dim QH^{4i}(X, Z/pZ) \leq 1$ for all $i$ and the equality holds at least for $1 \leq i \leq (p - 1)/2$.

Then $X_p = BSp_p^\otimes$ as an $H$-space.

A similar theorem can be formulated for $BSU$.

Again most of the proofs of Theorems 1.4 and 1.5 are in [4]. If $G = SU$ and $p$ is odd, Theorem 1.4 follows from Theorem 4.5 of [4] and if $p = 2$ the result follows as in the last part of the proof for odd $p$. Theorem 1.5 follows from Theorem 4.1 of [4]. Theorem 1.4 for $G = Sp$ and $p$ odd follows from Theorem 1.5. Therefore to complete the proof of Theorem 1.4 we must show that if $X_2 = BSp_2$, then $X_2 = BSp_2^\otimes$ as an $H$-space.

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2. The proofs of Theorems 1.1 and 1.3. We assume that $X \simeq BU$ at the prime $p$ and that the Frobenius map

$$\xi: H_2(X, Z/pZ) \to H_{2p}(X, Z/pZ)$$

is trivial. We must define an $H$-map $k: X_p \to BU_p^\otimes$ which is a homotopy equivalence. The key to the proof is Proposition 2.1 which is established using the main
We recall that \(T(x, r) \subset \mathbb{Q}[x]\) for \(0 \leq r \leq \infty\) is defined to be the subalgebra of \(\mathbb{Z}(p)\) generated by \(1, x, (p!)^{-1}x^p, \ldots, (p'!)^{-1}x^{p'}\). Analogously \(\Delta(x, r) \subset \mathbb{Q}[\xi]\) for \(0 \leq r \leq \infty\) is defined to be the subalgebra over \(\mathbb{Z}(p)\) generated by

\[1, \xi, (p!)^{-1}(\xi - 1) \cdots (\xi - p + 1), \ldots, (p'!)^{-1}(\xi - 1) \cdots (\xi - p' + 1).\]

We consider the zero dimensional complex \(K\)-homology group with coefficients \(\mathbb{Z}(p)\), the integers localized at the prime ideal \((p)\).

**Proposition 2.1.** (a) \(H_*(X, \mathbb{Z}(p)) \cong \Gamma(x_2, \infty) \otimes \mathbb{Z}(p)[x_4, x_6, \ldots, x_{2n}, \ldots]\) as rings, where both factors are sub-Hopf algebras.

(b) \(K_0(X, \mathbb{Z}(p)) \cong \Delta(\xi_2, \infty) \otimes \mathbb{Z}(p)[\xi_4, \xi_6, \ldots, \xi_{2n}, \ldots]\) as rings, where both factors are sub-Hopf algebras.

**Proof.** (a) Let \(A = H_*(X, \mathbb{Z}(p))\). Then by Theorem 1.1 of [3] we may write \(A = \bigotimes E(2r)\) as a Hopf algebra where \(r\) runs through those positive integers with \((r, p) = 1\) and as an algebra \(E(2r)\) is generated by classes with dimensions \(2p^r, i \geq 0\). Therefore \(A^* = \bigotimes E(2r)^* = H^*(X, \mathbb{Z}(p))\). Now \(X_p = BU_p\) and so the conditions of Theorem 3.1(c) of [4] are satisfied for \(r > 1\) and we deduce that \(E(2r)^* \cong \mathbb{C}(2r, 0)\) in the notation of that paper. But \(\mathbb{C}(2r, 0) \cong \mathbb{C}(2r, 0)\) as a Hopf algebra and as an algebra

\[\mathbb{C}(2r, 0) \cong \mathbb{Z}(p)[x_2r, x_2p, \ldots, x_{2p^r}, \ldots].\]

It remains to consider \(E(2)\). Theorem 1.1 of [5] implies that \(E(2) \cong \Gamma(x_2, r) \otimes D\) where \(D\) is a polynomial algebra. The condition on the Frobenius map implies that \(r \geq 1\) and so \(\dim Q^2 \{E(2) \otimes \mathbb{Z}/p\mathbb{Z}\} = 2\). Thus \(\dim P^2 \{(E(2) \otimes \mathbb{Z}/p\mathbb{Z})^*\} = 2\) and as \((E(2) \otimes \mathbb{Z}/p\mathbb{Z})^*\) is a polynomial algebra it follows that

\[\dim P^2 \{(E(2) \otimes \mathbb{Z}/p\mathbb{Z})^*\} \geq 2\]

for \(i = p^\alpha, \alpha \geq 1\).

Therefore \(\dim Q^2 \{E(2) \otimes \mathbb{Z}/p\mathbb{Z}\} \geq 2\) for \(i = p^\alpha, \alpha \geq 1\). It follows that \(E(2) \cong \Gamma(x_2, \infty) \otimes D\) and so

\[H_*(X, \mathbb{Z}(p)) \cong \Gamma(x_2, \infty) \otimes \mathbb{Z}(p)[x_4, x_6, \ldots, x_{2n}, \ldots]\]

as rings.

Clearly \(\Gamma(x_2, \infty)\) is a sub-Hopf algebra. By modifying the generators \(x_{2n}\) inductively using the basic technique of [3] and in particular of Lemma 2.2 and its proof, one can ensure that the polynomial algebra is a sub-Hopf algebra.

(b) Let \(h: X \to K(\mathbb{Z}(p), 2)\) realize a generator of \(H^2(X, \mathbb{Z}(p))\). Then \(h\) is an \(H\)-map and by part (a),

\[h_*: H_*(X, \mathbb{Z}(p)) \to H_*(K(\mathbb{Z}(p), 2), \mathbb{Z}(p)) \cong \Gamma(x_2, \infty)\]

is surjective. Therefore

\[h_0: K_0(X, \mathbb{Z}(p)) \to K_0(K(\mathbb{Z}(p), 2), \mathbb{Z}(p)) \cong \Delta(\xi_2, \infty)\]

is surjective and so is

\[h_0: K_0(X, \mathbb{Z}(p))_{2n} \to K_0(K(\mathbb{Z}(p), 2), \mathbb{Z}(p))_{2n}\]
in the CW-filtration for each $n$. Now by again modifying the generators $x_{2i}$ for $i > 1$ given in part (a), we can ensure that $h_\ast(x_{2i}) = 0$ for $i > 1$. We choose $\xi_{2i} \in K_0(X, \mathbb{Z}(p))_2$, whose image in

$$K_0(X, \mathbb{Z}(p))_{2i}/K_0(X, \mathbb{Z}(p))_{2i-2} \cong H_2(X, \mathbb{Z}(p))$$

is $x_{2i}$ and so again by modifying the original choices of the $\xi_{2i}$, we can ensure that $h_0(\xi_{2i}) = 0$ for $i > 1$. Therefore for each $i$,

$$(p !)^{-1}(\xi_{2i} - 1) \cdots (\xi_{2i} - p^i + 1) \in K_0(X, \mathbb{Z}(p))$$

and it follows that $K_0(X, \mathbb{Z}(p)) \cong \Delta(\xi_{2i}, \infty) \otimes \mathbb{Z}(p)[\xi_4, \xi_6, \ldots, \xi_{2n}, \ldots]$ as rings.

Again it is clear that $\Delta(\xi_{2i}, \infty)$ is a sub-Hopf algebra and by modifying the original $\xi_{2i}$ for $i > 1$ as in the similar step in (a) we can ensure that the polynomial algebra is a sub-Hopf algebra. This completes the proof.

It follows from Theorem 3.1 of [4] that the Hopf algebra structure of $H_\ast(X, \mathbb{Z}(p))$ is well defined. In fact $H_\ast(X, \mathbb{Z}(p)) \cong \Gamma(x_{2i}, \infty) \otimes (\mathbb{Z}(2,1) \otimes \cdots \otimes \mathbb{Z}(2r,0))$ where $r > 1$ and $(r, p) = 1$ as Hopf algebras. In particular $H_\ast(X, \mathbb{Z}(p)) \cong H_\ast(BU^\ast, \mathbb{Z}(p))$.

We can now complete the proofs. We recall [2, 6] that as an $H$-space $BU^\ast = K(\mathbb{Z}(p), 2) \times BSU_p^1 \times BU^2_p \times \cdots \times BU^{p-1}_p$ and that $H$-maps $g_i: X \to BU^i_p$ for $i > 1$ were constructed in §4 of [4] inducing monomorphisms of Hopf algebras

$$g_i^*: H^\ast(BU^i_p, \mathbb{Z}/p\mathbb{Z}) \to H^\ast(X, \mathbb{Z}/p\mathbb{Z}).$$

Therefore we must construct an $H$-map $g_i: X \to K(\mathbb{Z}(p), 2) \times BSU^1_p$ with similar properties, for then $g_1 \times g_2 \times \cdots \times g_{p-1}: X \to BU^\ast_p$ will be an $H$-equivalence.

The standard decomposition of $BU^\ast_p$ implies that $K_0(X, \mathbb{Z}(p))$ and $K^0(X, \mathbb{Z}(p))$ decompose canonically into $(p-1)$-summands and we assume that the generators $\xi_{2i}$ of Proposition 2.1(b) lie in their appropriate summands as in [4]. Now $\mathbb{Q}K_0(X, \mathbb{Z}(p)) \cong \mathbb{Q} \oplus \{ \otimes (\mathbb{Z}(p))^{\infty} \}$ where the $\mathbb{Z}(p)$ summands are generated by the images of the $\xi_{2i}$, (The $\mathbb{Q}$ on the right-hand side denotes the rationals [1].) Therefore

$$PK^0(X, \mathbb{Z}(p)) \cong \text{Hom}(\mathbb{Q}K_0(X, \mathbb{Z}(p)), \mathbb{Z}(p)) \cong \Pi(\mathbb{Z}(p))^{\infty}$$

generated by $\eta_{2i}$ for $i > 1$ such that $\eta_{2i}(\xi_{2i}) = \delta_{ij}$ and can be taken to lie in their appropriate summands. We consider $\eta_{2p} \in K^0(X, \mathbb{Z}(p))$. As it is primitive, it represents an $H$-map $\eta_{2p}: X \to BU^1_p$; it has exact CW-filtration $2p$ and the element in the associated graded group $H^{2p}(X, \mathbb{Z}(p))$ is a polynomial generator $\gamma_{2p}$. Now since $BU^1_p = K(\mathbb{Z}(p), 2) \times BSU^1_p$ as a space and $\eta_{2p} \in K^0(X, \mathbb{Z}(p))_{2p}$, $\eta_{2p}$ induces $\eta'_{2p}: X \to BSU^1_p$ with $i: BSU^1_p \to BU^1_p$ is the inclusion. As $i$ is an $H$-map, $\eta'_{2p}$ is an $H$-map. Further since $(\eta'_{2p})! : [BSU^1_p, BU] \to [X, BU]$ maps $i$ to $\eta_{2p}$ and $BSU^1_p$ is $(2p - 1)$-connected, the induced homomorphism

$$(\eta'_{2p})^*: QH^{2p}(BSU^1_p, \mathbb{Z}(p)) \to QH^{2p}(X, \mathbb{Z}(p))$$

is an isomorphism.

Let $h: X \to K(\mathbb{Z}(p), 2)$ realize a generator of $H^2(X, \mathbb{Z}(p))$. Then $g_1 = h \times \eta_{2p}^*$.

$$X \to K(\mathbb{Z}(p), 2) \times BSU^1_p$$

is an $H$-map and induces

$$g_1^*: QH^{-2i}(K(\mathbb{Z}(p), 2) \times BSU^1_p, \mathbb{Z}(p)) \to QH^{-2i}(X, \mathbb{Z}(p))$$
which is an isomorphism for $i = 1$ and $p$. Therefore as in the proof of Theorem 4.4 of [4], it induces an isomorphism for all $i$ of the form $1 + k(p - 1)$. Hence $k = g_1 \times g_2 \times \cdots \times g_p : X \to BU_p^\otimes$ is the mod $p$ $H$-equivalence we seek.

3. The proof of Theorem 1.4. Let $X \simeq_2 BS\rho$ where $X$ is a standard $H$-space. We must show that $X \simeq_2 BS\rho^\otimes$ as an $H$-space and to do this it is sufficient to construct an $H$-map $k : X \to BS\rho^\otimes$ which induces an isomorphism of cohomology groups in dimension 4.

**Lemma 3.1.** Let $k : X \to X$ induce an isomorphism $k^* : H^4(X, Z/2Z) \to H^4(X, Z/2Z)$. Then $k$ is a homotopy equivalence.

**Proof.** If we can show that $k^* : QH^4(X, Z/2Z) \to QH^4(X, Z/2Z)$ is an isomorphism for $i = 2$, then the action of the Steenrod algebra in $H^*(BS\rho, Z/2Z)$ ensures that $k^*$ is an isomorphism for $i > 2$. Thus $k^* : H^*(X, Z/2Z) \to H^*(X, Z/2Z)$ is an isomorphism. The lemma then follows from Whitehead's theorem.

The result needed in dimension 8 is most easily proved using complex $K$-theory. We know that

$$K^0(X, Z(2)) = Z(2)[[\eta_4, \eta_8, \ldots]]$$

where $\eta_{4n}$ has exact CW-filtration $4n$ and by direct and standard calculations we can choose $\eta_4$ and $\eta_8$ such that $\psi^2(\eta_4) = 4\eta_4 + 2\eta_8 + \eta_4^2$. We consider $k! : QK^0(X, Z(2)) \to QK^0(X, Z(2))$ and work mod $QK(X, Z(2))$. Then $k!(\eta_4) = \lambda\eta_4 + \mu\eta_8$ and $k!(\eta_8) = \nu\eta_8$. The hypotheses imply that $\lambda$ is a unit in $Z(2)$ and we wish to prove the same for $\nu$. Since $\psi^2(\eta_4) = 4\eta_4 + 2\eta_8$ and $\psi^2(\eta_8) = 16\eta_8$, from $\psi^2(k!\eta_4) = k!\psi^2(\eta_4)$ we deduce that $2\lambda = 2\nu \mod 4$ and so $\nu$ is a unit. This completes the proof of the lemma.

Theorem 3.1 and the comments which follow Corollary 3.2 in [4] imply that the Hopf algebra structure of $H_*(X, Z(2))$ is well defined. In the notation used there $H_*(X, Z(2)) = \otimes C(2r, 1)$, where $r$ runs over all positive odd integers. In particular,

$$H_*(X, Z(2)) \cong Z_2[\eta_4, \eta_8, \ldots]$$

and so $K_0(X, Z(2)) \cong Z_2[\xi_4, \xi_8, \ldots]$ and $QK_0(X, Z(2)) \cong \oplus (Z_2)^\infty$ where the $Z_2$ summands are generated by the images of the $\xi_{4i}$. Therefore

$$PK^0(X, Z(2)) = \text{Hom}(QK_0(X, Z(2)), Z_2) \cong \prod (Z_2)^\infty$$

generated by $\{\eta_{4i}\}$ where $\eta_{4i}(\xi_{4j}) = \delta_{i,j}$. Then $\eta_4 : X \to BU_2^\otimes$ is an $H$-map and regarded as an element of $K^0(X, Z(2))_4$ has image $\eta_4$ a generator in $H^4(X, Z(2))$. As $BU_2 = K(Z(2), 2) \times BSU_2$, $\eta_4^4 : H^4(BU_2, Z(2)) \to H^4(X, Z(2))$ is an isomorphism. We must show that $\eta_4^4 : X \to BU_2$ lifts to an $H$-map $\eta_4^4 : X \to BS\rho_2$ inducing an isomorphism of 4 dimensional cohomology groups.

We write $KO(\ )$ and $KH(\ )$ for real and symplectic zero dimensional cohomology groups with $Z(2)$-coefficients. Now $X_2$ and $X_2 \times X_2$ have local cellular structures in which all cells have dimension $4n$. Thus their real and symplectic $K$-groups
are torsion free and the complexification and restriction homomorphisms $c$: $\text{KO}(\ ) \to K^0(\ )$ and $c': \text{KH}(\ ) \to K^0(\ )$ are monomorphisms. We consider $\text{KO}(\ )$ and $\text{KH}(\ )$ as subgroups of $K^0(\ )$. We will require that for $X_2$ and $X_2 \times X_2$

(3.1) $K^0(\ ) = \text{KO}(\ ) + \text{KH}(\ )$ and $2K^0(\ ) = \text{KO}(\ ) \cap \text{KH}(\ )$.

For finite complexes with cells with dimensions $4n$, (3.1) can be proved by induction since $c: \text{KO}(S^{4k}, \mathbb{Z}(2)) \to K^0(S^{4k}, \mathbb{Z}(2))$ is an isomorphism if $k$ is even and has cokernel $\mathbb{Z}/2\mathbb{Z}$ if $k$ is odd and $c': \text{KH}(S^{4k}, \mathbb{Z}(2)) \to K^0(S^{4k}, \mathbb{Z}(2))$ is an isomorphism if $k$ is odd and has cokernel $\mathbb{Z}/2\mathbb{Z}$ if $k$ is even. The conclusion for $X_2$ and $X_2 \times X_2$ is obtained by taking inverse limits over the finite skeleta.

Now using (3.1) we write $\eta_4 = r + q$ where $r \in \text{KO}(X, \mathbb{Z}(2))$ and $q \in \text{KH}(X, \mathbb{Z}(2))$. We consider $\tilde{m}!: K^0(X, \mathbb{Z}(2)) \to K^0(X \times X, \mathbb{Z}(2))$ where $\tilde{m}! = m! - \pi_1! - \pi_2!$ so that $\tilde{m}!(\eta_4) = 0$. Thus $\xi = \tilde{m}!(r) = -\tilde{m}!(q)$ is both real and symplectic and therefore by (3.1) $\xi = 0$ mod 2. Now the Hopf algebra structure of $H^*(X_2, \mathbb{Z}(2)) \cong \bigotimes C(2r, 1)$ is well understood and $\text{PH}^*(X_2, \mathbb{Z}(2)) \cong \bigoplus (\mathbb{Z}(2))^\infty$ generated by $y_{2i}$ elements in the associated graded group corresponding to $\eta_{4i}$. In $H^*(X_2, \mathbb{Z}(2))$ if a homogeneous element $z$ is primitive mod 2, then $z = z' + 2z''$ where $z'$ is primitive.

By working inductively along the filtration in $K^0(X_2, \mathbb{Z}(2))$ one deduces that if $\tilde{m}!(\phi) = 0$ mod 2, then $\phi = \phi' + 2\phi''$ where $\tilde{m}!(\phi') = 0$. We deduce that $q = q' + 2q''$ where $q' \in K^0(X_2, \mathbb{Z}(2))$ is primitive and as $q' = q - 2q''$ it lies in $KH(X_2, \mathbb{Z}(2))$.

Let $r' = r + 2q''$. Then $\eta_4 = q' + r'$ where $q': X_2 \to BS\mathcal{P}_2$ is an H-map. Now $r' \in K^0(X_2, \mathbb{Z}(2))_4$ has an image in $H^4(X_2, \mathbb{Z}(2))$ which is divisible by 2, since $\text{KO}(S^4) \to K^0(S^4)$ has cokernel $\mathbb{Z}/2\mathbb{Z}$. Thus the image of $q'$ in $H^4(X_2, \mathbb{Z}(2))$ is a generator.

We set $k = q': X_2 \to BS\mathcal{P}_2$ which has all the properties required of it to complete the proof.

**REFERENCES**