A NECESSARY AND SUFFICIENT CONDITION FOR A CONNECTED AMENABLE GROUP TO HAVE POLYNOMIAL GROWTH

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ABSTRACT. It is shown that a connected amenable group $G$ has polynomial growth if, and only if, given any open subsemigroup $S$ of $G$ and a compact set $K$ in $G$ there exists an $s$ in $S$ such that $Ks \subseteq S$.

Introduction. Let $G$ be a connected topological group and $C(G)$ the space of all bounded continuous real-valued functions on $G$. For each $f$ in $C(G)$ and $a$ in $G$ define $\|f\| = \sup\{|f(x)|: x \in G\}$ and $l_af(t) = f(a^{-1}t)$ for all $t$ in $G$. A function $f$ in $C(G)$ is left uniformly continuous if whenever $(a_i)$ is a net in $G$ and converges to some $s$ in $G$, then $\|l_a f - l_s f\|$ tends to zero. We say that $G$ is amenable (more accurately, left amenable) if there exists an $m$ in $LUC(G)^*$, the conjugate space of $LUC(G)$ (the space of left uniformly continuous functions), such that $m \geq 0$, $\|m\| = 1$ and $m(l_a f) = m(f)$ for all $f$ in $LUC(G)$ and $a$ in $G$.

The reader is referred to Greenleaf [3] for many interesting properties of amenable groups. Another interesting class of groups are those with polynomial growth, whose definition follows. Let $|A|_G$ denote the left Haar measure of a measurable subset $A$ of $G$. A locally compact group $G$ is said to have polynomial growth if for each compact nbd $U$ of the identity $e$ in $G$, there exists a polynomial $p$ such that $|U^n|_G \leq p(n)$ for all $n = 1, 2, 3, \ldots$.

The notion of Archimedean property for groups was introduced in [2]. A group $G$ is said to have Archimedean property if whenever $S$ is an open generating subsemigroup and $K$ a compact subset of $G$ there exists an $s$ in $S$ such that $sK \subseteq S$. It is proved in [2] that any connected group with polynomial growth has Archimedean property. Some remarkably diverse properties of groups with polynomial growth can be found in [1, 4, 7]. It is well known that groups with polynomial growth are amenable. However, the converse fails. The group of affine transformations on the real line is a connected solvable group without polynomial growth. Jenkins [6] has given a necessary and sufficient condition for a connected amenable group to have polynomial growth. In this paper we give another necessary and sufficient condition in Theorem 2. As a corollary we note that every bounded continuous function on a group with polynomial growth which is left uniformly continuous on a dense open subsemigroup is left uniformly continuous on $G$.

First we prove the following

PROPOSITION 1. Let $G$ be a connected locally compact group and $S$ an open subsemigroup of $G$. Then the following are equivalent.

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(a) The open right ideals of $S$ have finite intersection property, i.e., any finite set of nonempty open right ideals of $S$ has nonempty intersection.

(b) $G = SS^{-1}$.

(c) Given any compact set $K$ in $G$ there exists an $s$ in $S$ such that $Ks \subseteq S$.

**Proof.** (b) implies (c). First, we prove that given any finite subset $F$ of $G$ there exists a $t$ in $S$ such that $Ft \subseteq S$. If $F = \{k_1, k_2, \ldots, k_n\}$ there exists an $s_1$ in $S$ such that $k_1s_1$ is in $S$; there exists an $s_2$ in $S$ with $(k_2s_1)$ in $S$, etc. Then $F_s_1s_2 \cdots s_n$ is a subset of $S$. Now, if $K$ is any compact subset (given) fix an $s$ in $S$. Let $U = U^{-1}$ be a nbd of $e$ with compact closure such that $A = sU$ is a subset of $S$. Consider the compact set $s^{-1}K$; since $G$ is connected, there is a large $N$ such that $s^{-1}K \subseteq U^N$. Since $U$ is an open set with compact closure there exists a finite set $F$ such that $UU \subseteq UV \subseteq UF$ and, choosing $t$ as above, $U^Nt \subseteq UF^{N-1}t \subseteq US$. Hence, $Kt \subseteq s^Nt \subseteq sUS \subseteq SS \subseteq S$, which proves (c).

The proof of (c) implies (b) is obvious.

(b) implies (a). Suppose $I$ and $J$ are two open right ideals of $S$ which are disjoint. Consider $b^{-1}a$ in $G$ where $a \in I$, $b \in J$; then $b^{-1}a = s_1s_2^{-1}$ for some $s_1$ and $s_2$ in $S$. Hence, $(bs_1)^{-1} = s_1^{-1}b^{-1} = s_2^{-1}a^{-1} = (as_2)^{-1}$, so that $I^{-1} \cap J^{-1} \neq \emptyset$, a contradiction.

(a) implies (b). $SS^{-1}$ is an open subgroup of $G$ containing $e$, the identity of $G$. For, if $a, b, c$ and $d$ are in $S$ then $ab^{-1}cd^{-1} \in SS^{-1}$. Since $cS$ and $bS$ are open right ideals in $S$, $cS$ and $bS$ have an element in common, i.e., there exists $s_1, s_2$ in $S$ such that $c_s_1 = bs_2$, which implies $ab^{-1}cd^{-1} = as_2(ds_1)^{-1} \in SS^{-1}$. This shows that $SS^{-1}$ is an open subgroup and hence closed. Since $G$ is connected, $G = SS^{-1}$.

With very few modifications to [2, Lemma 7] one can show that any connected locally compact group with polynomial growth has property (c) of the above proposition. Also, note that $G = S^{-1}S$ for any open subsemigroup $S$ of $G$ is equivalent to: given any compact set $K$ in $G$ there exists an $s$ in $S$ such that $sK \subseteq S$. The above two remarks imply that if $G$ has polynomial growth then for any open subsemigroup $S$ of $G$, $G = SS^{-1} = S^{-1}S$.

Jenkins [5, 6] has shown that a connected amenable group $G$ has polynomial growth if, and only if, each open subsemigroup of $G$ is amenable if, and only if, the open right ideals of an open subsemigroup of $G$ have finite intersection property. Using the above remark and the Proposition we have

**Theorem 2.** Let $G$ be a connected group. Then the following are equivalent.

(a) $G$ is an amenable group with the property that given any compact set $K$ in $G$ and any open semigroup $S$ of $G$, then there exists an $s$ in $S$ such that $Ks \subseteq S$.

(b) $G$ has polynomial growth.

The following is worth noting.

**Corollary.** Let $G$ be a connected group with polynomial growth and $S$ an open dense subsemigroup of $G$. If $F$ is a bounded continuous function on $G$ such that $F|_S$ is in $LUC(S)$, then $F$ is in $LUC(G)$.

**Proof.** Since $G$ has polynomial growth, it is an Archimedean group and $G = SS^{-1}$. The rest of the proof follows from Lau [8].

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