PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 2, February 1985

A GENERALIZED JACOBIAN CRITERION FOR REGULARITY

PETER SEIBT

ABSTRACT. For a commutative noetherian algebra B over a perfect field A regularity is equivalent to the flatness of $\Omega_{B|A}$ plus $H_1(A, B, B) = 0$ (simplicial homology). In characteristic 0 the homological condition is superfluous.

1. Introduction. All rings to be considered will be commutative and unitary. The classical Jacobian criterion (in terms of Kähler differentials) reads like this: Let A be a perfect field, $(B, \mathfrak{m}, K)$ a local A-algebra of finitely generated type (obtained by localizing an A-algebra of finite type) such that $K = B/\mathfrak{m} = A$ and let $\Omega_{B|A}$ be the B-module of A-differentials for B. Then B is regular if and only if $\Omega_{B|A}$ is free (of rank equal to the dimension of B).

I want to generalize this differential characterization of regularity in the following way.

THEOREM. *Let A be a perfect field, B a noetherian A-algebra. Then B is regular (all localizations $B_{\mathfrak{m}}$ of B at maximal ideals $\mathfrak{m}$ are regular local rings) if and only if the following two conditions hold:*

(1) *$\Omega_{B|A}$ is B-flat.*

(2) *For every infinitesimal A-extension*

$$E = 0 \to M \to C \to B \to 0 \quad \text{of } B,$$

the associated derived module sequence

$$\operatorname{diff}(E) = 0 \to M \to \Omega_{C|A} \otimes_C B \to \Omega_{B|A} \to 0$$

is exact.

COROLLARY. *If* $\operatorname{char} A = 0$, *we have*

$$B \text{ is regular if and only if } \Omega_{B|A} \text{ is } B\text{-flat}.$$

I do not know whether condition (2) is really relevant in positive characteristic (A perfect, of course).

2. Proof of the theorem. Recall for arbitrary $A \to B$ the two sequences $(H_n(A, B, -))_{n \geq 0}$ and $(H^n(A, B, -))_{n \geq 0}$ of simplicial homology and cohomology functors as defined and discussed in **[1]**.

Received by the editors February 13, 1984.
1980 *Mathematics Subject Classification*. Primary 13D03, 13H05.
Key words and phrases. Regular local ring, differentials, simplicial (co)homology.

LEMMA. *For arbitrary $A \to B$ the following three conditions are equivalent*:
(a) $\mathrm{Ext}^1_B(\Omega_{B|A}, -) = H^1(A, B, -)$.
(b) $H_1(A, B, B) = 0$.
(c) $H_1(A, B, -) = \mathrm{Tor}_1^B(\Omega_{B|A}, -)$.

Condition (a) needs some comment:

For every B-module M, $\mathrm{Ext}^1_B(\Omega_{B|A}, M)$ classifies the singular 1-extensions of $\Omega_{B|A}$ by M, whereas $H^1(A, B, M)$ classifies the infinitesimal A-extensions of B by M. There is a B-monomorphism $\mathrm{Ext}^1_B(\Omega_{B|A}, M) \to H^1(A, B, M)$, functorial in M, whose image consists of the classes of those A-extensions $E = 0 \to M \to C \to B \to 0$, which have an exact derived module sequence

$$\mathrm{diff}(E) = 0 \to M \to \Omega_{C|A} \otimes_C B \to \Omega_{B|A} \to 0$$

(see [**3**, pp. 158–161]). Thus (a) is only another formulation of condition (2) of the theorem.

PROOF OF THE LEMMA. (a) implies (b). We have $H^1(A, B, W) = 0$ for every injective B-module W. But for injective B-modules

$$H^1(A, B, W) = \mathrm{Hom}_B(H_1(A, B, B), W)$$

by [**1**, 3.21, p. 42]. Now take W the injective hull of $H_1(A, B, B)$.

(b) implies (c). This follows from [**1**, 3.19, p. 41].

(c) implies (a). We have immediately $H_1(A, B, B) = 0$, and thus $H^1(A, B, W) = 0$ whenever W is B-injective (see (a) $\Rightarrow$ (b)). But by [**2**], $H^1(A, B, -)$ is part of an exact connected sequence of cohomology functors, say $(D^n(B|A, -))_{n \geqslant 0}$, where $D^0(B|A, -) = \mathrm{Hom}_B(\Omega_{B|A}, -)$, $D^1(B|A, -) = H^1(A, B, -)$ and $D^n(B|A, -) = \mathrm{Ext}^{n-1}_B(K, -)$, $n \geqslant 2$, for some B-module K. The vanishing on injective coefficients in positive degree yields, in particular, $H^1(A, B, -) = \mathrm{Ext}^1_B(\Omega_{B|A}, -)$.

This finishes the proof of the lemma.

Now the theorem follows easily:

By the lemma (condition (b)) we may assume B to be local. Since A is perfect, the regularity of B is equivalent to the formal smoothness of B over A. But by [**1**, Theorem 30, p. 331] this in turn is equivalent to $H_1(A, B, -) = 0$.

The lemma now immediately yields the assertion of the theorem.

As to the corollary, we may rest in the local case. The flatness of $\Omega_{B|A}$ gives the $B|\mathfrak{m}^k$-projectivity of $\Omega_{B|A} \otimes_B B/\mathfrak{m}^k$ for every $k \geqslant 1$ ($\mathfrak{m}/\mathfrak{m}^k$ is nilpotent, $\mathfrak{m}$ the maximal ideal of B, of course).

A theorem of Radu's (cf. [**1**, 7.31, p. 103]) guarantees the regularity of B.

REFERENCES

1. M. André, *Homologie des algèbres commutatives*, Springer-Verlag, Berlin, Heidelberg and New York, 1974.

2. M. Barr, *A cohomology theory for commutative algebras*. I, II, Proc. Amer. Math. Soc. **16** (1965), 1379–1391.

3. L. Illusie, *Complexe cotangent et déformations*. I, Lecture Notes in Math., vol. 239, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

U. E. R. DE MATHEMATIQUES ET D'INFORMATIQUE, UNIVERSITÉ DE BORDEAUX I, F-33405 TALENCE, FRANCE