GENERALIZED INTERSECTION MULTIPlicITIES
OF MODULES. II

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ABSTRACT. The vanishing conjecture for intersection multiplicities in dimensions \( \leq 5 \) is true in greater generality than previously known.

In this note we prove a more general theorem than Corollary 2.5 of [D1]. Since both the vanishing and nonvanishing conjectures are not true in general [DHM], it is now important to understand the following question: What can we say about

\[
\chi^R(M, N) = \sum_{i=0}^{d} (-1)^i l(\text{Tor}_i^R(M, N)),
\]

where \( R \) is a Gorenstein ring, and \( M \) and \( N \) are modules over \( R \) with finite projective dimension such that \( l(M \otimes_R N) < \infty \), \( \dim M + \dim N \leq \dim R \), and \( d = \text{p.d.} M \)?

The following theorems partially answer the question up to dimension 5.

Throughout §1 all tensor products, Tor’s and Euler characteristics are computed over \( R \).

1.1. Theorem. Let \( R \) be a Gorenstein ring of dimension \( \leq 5 \). Let \( M \) and \( N \) be modules of finite projective dimension with \( l(M \otimes_R N) < \infty \) and \( \dim M + \dim N < \dim R \). Then \( \chi(M, N) = 0 \).

Proof. First we note that §§1.1 and 1.3 of [D1] hold true in the above case. The sufficient part of Theorem 2.4 of [D1] is still valid, though the statement has to be changed a bit in the following way:

1.2. Theorem. Let \( R \) be a Gorenstein ring of dim \( N \). If for any two perfect modules \( M \) and \( N \) with (i) \( \dim M + \dim N = \dim R \), and (ii) \( l(M \otimes_R N) < \infty \), \( l(M \otimes N) = l(M \otimes \tilde{N}) \), then the vanishing conjecture holds in \( R \) for every pair of modules of finite projective dimension. Here \( \tilde{N} = \text{Ext}^{\dim N-1}(N, R) \), \( r = \dim N \).

Proof of this theorem can be obtained by following the argument of (2.4) of [D1].

To prove our theorem, by 1.2 it is enough to prove that

(1) \( l(T \otimes_R Q) = l(T \otimes_R \tilde{Q}) \ldots \),

where \( T \) and \( Q \) are any two perfect modules over \( R \) with \( l(T \otimes Q) < \infty \) and \( \dim T + \dim Q = \dim R \).

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Case 1. \( \dim T = 5 \), i.e., \( T \) is free, hence \( l(Q) < \infty \). Since \( l(Q) = l(\hat{Q}) \), \( l(T \otimes Q) = l(T \otimes \hat{Q}) \).

Case 2. \( \dim T = 4 \). In this case by killing an \( R \)-sequence which is also an \( M \)-sequence of length four contained in \( \text{Ann}_R Q \), we reduce the proof to proving (1) on a Gorenstein ring of dimension 1 and, hence, we are done.

Case 3. \( \dim T = 3 \). In this case killing an \( M \)-sequence which is also an \( R \)-sequence of length 3, contained in \( \text{Ann}_R Q \), we reduce the proof to proving (1) on a Gorenstein ring of dim 2 and we are done by (1.1) of [D1].

Similar arguments hold true when \( \dim R \leq 4 \).

The next theorem throws a little more light on the nonvanishing conjecture. (This is a changed version of Proposition (3.4) of [D1].)

1.3. Theorem. Let \( R \) be a complete Gorenstein ring of \( \dim 5 \) whose coefficient ring is a discrete valuation ring \( V \), and let \( p \) be a generator of the maximal ideal of \( V \). Let \( M \) and \( N \) be two modules over \( R \) such that (i) \( M \) is perfect and \( \text{p.d. } N < \infty \), (ii) \( l(M \otimes_R N) < \infty \), (iii) \( \dim M + \dim N = 5 \), (iv) \( pN = 0 \), and \( p \) is a n.z.d. on \( M \). Then \( \chi(M, N) > 0 \).

For the proof we refer the reader to (1.8) and Corollaries 5 and 6 of [D2].

Remark. Theorems (2.2) and (2.4) of [D1] and Theorem 1.1 can be proved with the assumption that \( R \) is a Cohen-Macaulay ring. The techniques are the same; one has only to consider spectral sequences \( \text{Ext}^r_S(\text{Tor}^R_r(M, N), S) \) and \( \text{Ext}^r_R(M, \text{Ext}^r_S(N, S)) \), and instead of taking \( \hat{M}, \hat{N} \) as defined in the Gorenstein case with \( \text{p.d. } M = d \), say, one should consider \( \hat{M} = \text{Ext}^r_S(M, R) \) and \( \hat{N} = \text{Ext}^r_S(N, S) \), \( r = \dim N \). Here \( S \) is a Gorenstein ring such that \( R = S/I \) and \( \dim R = \dim S \) (assuming \( R \) to be complete).

References


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