SELF-DUAL LATTICES FOR MAXIMAL ORDERS
IN GROUP ALGEBRAS
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ABSTRACT. Let $G$ be a finite group and $V$ an irreducible $\mathbb{Q}[G]$-module. Let $R$ be a Dedekind domain with quotient field $\mathbb{Q}$ such that $|G|$ is a unit in $R$. For applications to topology it is of interest to know if $V$ contains a full self-dual $R[G]$-lattice. We show that such lattices always exist for some major classes of finite groups.

Let $G$ be a finite group and let $R$ be a Dedekind domain with quotient field $\mathbb{Q}$ such that $|G|$ is a unit in $R$. We say that a $\mathbb{Q}[G]$-module $V$ is balanced if $V$ contains a full self-dual $R[G]$-lattice. For applications to surgery theory (see [10, p. 28–36]) it is of interest to establish criteria for balance.

We show that any irreducible $\mathbb{Q}[G]$-module is balanced when $G$ is $p$-hyperelementary for an odd prime $p$, when $|G|$ is odd, or when $G$ is a 2-group. We know of no example of an unbalanced $\mathbb{Q}[G]$-module for any finite group $G$. Theorem 3, our main criterion for balance, follows easily from standard but deep results in integral representation theory.

I would like to thank Bruce Williams for bringing this problem to my attention.

Preliminaries. For $G$ and $R$ as above, $R[G]$ is a maximal order in $\mathbb{Q}[G]$ by [9, Theorem 41.1]. Therefore every left $R[G]$-lattice $L$ is projective, and $L$ is indecomposable if and only if $\mathbb{Q}L$ is an irreducible $\mathbb{Q}[G]$-module [9, Corollary 21.5]. The central primitive idempotents $e_i$ ($1 \leq i \leq m$) of $\mathbb{Q}[G]$ lie in $R[G]$ by [9, Theorem 10.5], and any $R[G]$-lattice $L$ decomposes as $L = e_1L \oplus \cdots \oplus e_mL$. If $L$ and $M$ are isomorphic $R[G]$-lattices, so are $e_iL$ and $e_iM$ for each $i$. By [9, Theorems 11.1 and 18.7] two $R[G]$-lattices $L$ and $M$ belong to the same genus if and only if $QL \cong QM$. If $H \leq G$ and $L$ is an $R[H]$-lattice, then $L^G$ denotes the induced lattice $R[G] \otimes_{R[H]} L$.

If $L$ is a left $R[G]$-lattice, then $L^*$ denotes the dual (contragredient) left $R[G]$-lattice. If $L$ affords the matrix representation $\rho: G \to \text{GL}(n, R)$, then $L^*$ affords the composition of $\rho$ with the inverse transpose automorphism of $\text{GL}(n, R)$. In particular, $L \cong L^{**}$. For $e_i$ as above, $e_iL^* \cong (e_iL)^*$. If $L$ is an $R[H]$-lattice for some $H \leq G$, then $(L^*)^G \cong (L^G)^*$.

If $V$ is a $\mathbb{Q}[G]$-module, let $\chi_V$ denote the character of $V$. If $\chi$ is an irreducible complex character of $G$, let $m(\chi)$ denote the Schur index of $\chi$ over $\mathbb{Q}$, and let $\text{Tr}(\chi)$ denote the sum of the distinct algebraic conjugates of $\chi$. If $\psi$ is any rational-valued character of $G$, let $p(\psi)$ be the permutation index of $\psi$—the least integer $p$ such that $p\psi$ is an integral linear combination of permutation characters of $G$. By [4, Theorem 5.21] $p(\psi)$ divides $|G|$.

Received by the editors April 5, 1984.

1980 Mathematics Subject Classification. Primary 20C10; Secondary 20C15.

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The following lemmas contain the key results we need from integral representation theory. If $M$ and $N$ are $R[G]$-lattices, we write $M|N$ to mean that $M$ is isomorphic to a direct summand of $N$.

**Lemma 1.** Let $V$ be an irreducible $Q[G]$-module. Let $M$ and $N$ be $R[G]$-lattices such that $QM \cong mV$ and $QN \cong nV$, with $m < n$. Then $M|N$.

**Proof.** Let $M = M_1 \oplus \cdots \oplus M_m$ and $N = N_1 \oplus \cdots \oplus N_n$ be decompositions of $M$ and $N$ as direct sums of indecomposable lattices. Since $M_1, N_1,$ and $N_2$ belong to the same genus, we may write $N_1 \oplus N_2 \cong M_1 \oplus L_1$ for an indecomposable $R[G]$-lattice $L_1$ by [9, Corollary 27.3]. Let $M' = M_2 \oplus \cdots \oplus M_m$ and let $N' = L_1 \oplus N_3 \oplus \cdots \oplus N_n$. Then $M \cong M_1 \oplus M'$ and $N \cong M_1 \oplus N'$. By induction on $m$ we may assume $M' | N'$. Hence $M | N$. \[\square\]

We say that a $Q[G]$-module $V$ is Eichler if no simple component of $\text{End}_{Q[G]}[V]$ is a totally definite quaternion algebra, as defined in [9, p. 293]. We note that if $V$ is an irreducible $Q[G]$-module which is not Eichler, and $\chi$ is an irreducible complex constituent of $\chi_V$, then $\chi(1) = m(\chi) = 2$. The structure of $G/Ker V$ is severely restricted; see [9, p. 344]. We say that an $R[G]$-lattice $L$ is Eichler if $QL$ is Eichler.

**Lemma 2.** Let $X, M$, and $N$ be $R[G]$-lattices. If $X \otimes M \cong X \otimes N$ and $M$ is Eichler, then $M \cong N$.

**Proof.** See [5, p. 14]. \[\square\]

**The main criterion.**

**Theorem 3.** Let $V$ be an irreducible $Q[G]$-module. Suppose $p(\chi_V)$ is odd and $V$ is Eichler. Then $V$ is balanced.

**Proof.** Let $p = p(\chi_V)$. Then $pV \oplus V_1 \cong V_2$ for $Q[G]$-permutation modules $V_1$ and $V_2$. Let $L_1$ and $L_2$ be full self-dual $R[G]$-lattices in $V_1$ and $V_2$, respectively. Let $e$ be the primitive central idempotent in $Q[G]$ which corresponds to $V$. Then $eL_1$ and $eL_2$ are full self-dual $R[G]$-lattices in $eV_1$ and $eV_2$, respectively. By Lemma 1 we may write $eL_2 = eL_1 \oplus L_0$ for an $R[G]$-lattice $L_0$ with $QL_0 \cong pV$. Taking duals yields $eL_2 \cong eL_1 \oplus L_0^\circ$. By Lemma 2 we have $L_0 \cong L_0^\circ$.

Now let $M$ be a fixed $R[G]$-lattice with $QM \cong V$. Let $M_0 = \frac{1}{2}(p - 1)(M \oplus M^*)$. By Lemma 1 we may write $L_0 \cong M_0 \oplus M_1$, where $QM_1 \cong V$. Since $M_0^\circ \cong M_0$ and $L_0^\circ \cong L_0$, we have $L_0 \cong M_0 \oplus M_1^*$. Lemma 2 yields $M_1 \cong M_1^*$. \[\square\]

We recall that a group $G$ is called $p$-hyperelementary if $G$ has a cyclic normal $p$-complement.

**Corollary 4.** Suppose $G$ is $p$-hyperelementary for an odd prime $p$, $|G|$ is odd, or $G$ is abelian. Then every irreducible $Q[G]$-module $V$ is balanced.

**Proof.** Suppose first that $G$ is $p$-hyperelementary for an odd prime $p$. Let $\chi$ be an irreducible complex constituent of $\chi_V$. By [4, Theorem 6.15] $\chi(1)$ is odd, so $V$ is Eichler. By [6, Definition 1.6 and Proposition 7.2] $\chi_V$ has odd permutation index. Hence, $V$ is balanced by Theorem 3. A similar argument works if $|G|$ is odd.

If $G$ is cyclic, then $G$ is $p$-hyperelementary for any odd prime $p$, so $V$ is balanced. If $G$ is abelian, then $G/Ker V$ is cyclic, so $V$ is balanced. \[\square\]
COROLLARY 5. Let $G$ be a finite group and $V$ an irreducible $\mathbb{Q}[G]$-module. If $V$ is Eichler and $V_H$ is balanced for every 2-hyperelementary subgroup $H$ of $G$, then $V$ is balanced.

PROOF. Let $\mathcal{H}$ be the family of all hyperelementary subgroups of $G$. By [4, Theorem 8.10] we may write

$$1_G = \sum_{H \in \mathcal{H}} a_H 1_H^G - \sum_{H \in \mathcal{H}} b_H 1_H^G,$$

where all the $a_H$ and $b_H$ are nonnegative integers. Then

$$\chi_V = \sum_{H \in \mathcal{H}} a_H (\chi_V|_H)^G - \sum_{H \in \mathcal{H}} b_H (\chi_V|_H)^G.$$

Hence,

$$V \oplus \bigoplus_{H \in \mathcal{H}} b_H (V_H)^G \cong \bigoplus_{H \in \mathcal{H}} a_H (V_H)^G.$$

By the hypotheses and Corollary 4, $V_H$ and, hence, $(V_H)^G$ are balanced for all $H \in \mathcal{H}$. Let $L_1$ and $L_2$ be full self-dual $\mathbb{R}[G]$-lattices in $\bigoplus_{H \in \mathcal{H}} b_H (V_H)^G$ and $\bigoplus_{H \in \mathcal{H}} a_H (V_H)^G$, respectively. The argument in the first paragraph of the proof of Theorem 3 shows that $V$ is balanced. ■

REMARKS. When $G$ is 2-hyperelementary and $V$ is an irreducible $\mathbb{Q}[G]$-module, there is a subgroup $H$ of $G$ and a primitive $\mathbb{Q}[H]$-module $W$ such that $V = W^G$. Let $H = H/\ker W$. Since every normal abelian subgroup of $H$ is cyclic, an application of [11, Lemma 2.3] to $H^2(x)$ shows that $H$ contains a self-centralizing normal cyclic subgroup. Thus the question of whether $V$ is balanced reduces in a sense to a Galois action situation, as in the Brauer-Witt theorem on Schur indices; see [4, Theorem 10.7].

We also remark that Corollaries 4 and 5 do not exhaust the applications of Theorem 3. See [2, 7 and 8] for more information about permutation indices.

2-groups. We prove a strong form of the balance property for 2-groups.


PROOF. Let $G$ be a 2-group with a faithful irreducible primitive $\mathbb{Q}[G]$-module $V$. To prove the proposition, it suffices to show that $|G| \leq 2$. Since $G$ has no noncyclic normal abelian subgroup, [3, Theorem 5.4.10] shows that $G$ is cyclic, dihedral, semidihedral, or generalized quaternion. Also, $G \neq D_8$ and we may assume $G \neq 1$.

Suppose $G$ is not cyclic and $|G| > 8$. Let $< x >$ be the maximal cyclic subgroup of $G$ and choose $t \in G$ so that $G = < t, x >$. If $G$ is not generalized quaternion, choose $t$ to be an involution. Let $G_0 = < t, x^2 >$. Let $\lambda$ be a faithful linear character of $< x >$ and let $\lambda(x) = \epsilon$. Then $\epsilon$ is a primitive $2^n$th root of 1 for some $n \geq 3$. Let $\chi = \lambda^G$ and let $\chi_0 = \chi|_{G_0} = (\lambda^G|_{< x^2 >})^{G_0}$. Then $\chi$ and $\chi_0$ are irreducible complex characters of $G$ and $G_0$, respectively.

The field of values $\mathbb{Q}(\chi_0)$ is contained in $\mathbb{Q}(\epsilon^2)$, while $\chi(x) = \epsilon + \epsilon^{-1}$ or $\epsilon + \epsilon^{2^{n-1}-1}$. Let $\sigma$ be the unique nonidentity field automorphism of $\mathbb{Q}(\epsilon)$ which fixes $\epsilon^2$. Then $\epsilon^\sigma = \epsilon^{2^{n-1}+1} = -\epsilon$. Hence, $\chi(x)^\sigma = -\chi(x) \neq 0$, so that $\chi^\sigma \neq \chi$, $\chi_0^G = \chi + \chi^\sigma$, and $[\mathbb{Q}(\chi) : \mathbb{Q}(\chi_0)] = 2$. By [1, 11.7 and 11.8] we have $m(\chi) = m(\chi_0)$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Therefore \((m(\chi_0)\text{Tr} \chi_0)^G = m(\chi)\text{Tr}(\chi)\). It follows that \(V\) is induced from an irreducible \(\mathbb{Q}[G_0]\)-module, contrary to assumption.

Thus \(G = Q_8\) or \(G\) is cyclic. In these cases \(G\) has a unique faithful irreducible \(\mathbb{Q}[G]\)-module which is induced from the unique subgroup of \(G\) of order 2. Since \(V\) is primitive, \(|G| = 2\), \(\blacksquare\).

**FINAL REMARK.** The referee has pointed out that it would be more significant to discuss whether a \(\mathbb{Q}[G]\)-module contains a self-dual \(\mathbb{Z}[G]\)-lattice rather than a self-dual \(\mathbb{R}[G]\)-lattice. For the application to topology, however, our results on \(\mathbb{R}[G]\)-lattices are significant. They show that, for appropriate \(G\), the computation of relative surgery obstruction groups arising in a certain long exact sequence reduces to the computation of surgery obstruction groups for maximal orders in division algebras.

**REFERENCES**