LOCAL UNCERTAINTY INEQUALITIES FOR FOURIER SERIES

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Abstract. Necessary and sufficient conditions are given on \( \alpha, \beta \) and \( t \) for there to exist a constant \( K \) such that

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K|E|^\alpha \|f|t\|,
\]

for all \( f \in L^1(T^d) \) and finite \( E \subset \mathbb{Z}^d \).

1. Introduction. The classical uncertainty principle inequality [2] states that

\[
\| |x|f(t)\|_2 \| |y|F\|_2 \geq (d/4\pi) \| f \|_2^2
\]

for all functions \( f \in L^2(\mathbb{R}^d) \), where the Fourier transform \( F \) is defined by

\[
F(y) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi i x \cdot y) \, dx \quad \text{for } y = (y_1, \ldots, y_d) \in \mathbb{R}^d.
\]

The natural generalization of this to a product \( T^d = (\mathbb{R}/2\pi \mathbb{Z})^d \) of circle groups fails since \( \sum_{n \in \mathbb{Z}^d} |n|^2 |\hat{f}(n)|^2 = 0 \) for all constant functions \( f \in L^2(T^d) \). (Here and below,

\[
\hat{f}(n) = (2\pi)^{-d} \int_{T^d} f(x) \exp(-in \cdot x) \, dx \quad \text{for } n = (n_1, \ldots, n_d) \in \mathbb{Z}^d.
\]

Recently, local versions of (1.1) have been developed and applied in quantum physics [3, 4]. Roughly speaking, they assert that if a function is condensed, then not only is its Fourier transform broad, but it cannot be "too" localized. Some of these inequalities involve general \( L^p \)-norms and powers of \( |x| \). For example, in [4] it is shown that, given \( t \in [1, \infty) \) and \( \beta \in \mathbb{R} \), there exists a constant \( K \) such that

\[
\left( \int_E |F(y)|^2 \, dy \right)^{1/2} \leq Km(E)^{\beta-d/t^*} \|f|\beta\|_t
\]

for all \( f \in L^2(\mathbb{R}^d) \) and measurable \( E \subset \mathbb{R}^d \), with \( m(E) < \infty \) if and only if \( d/t^* < \beta < d/t' \) or \( (t, \beta) = (1, 0) \) or \( (2, 0) \), where \( t' = t/(t - 1) \) and \( t^* = 2t/(t - 2) \). Furthermore, no other power of \( m(E) \) apart from \( \beta - d/t^* \) is possible. (Global extensions of (1.1) are given in [1].)

Here we establish the following analogous result for \( T^d \).
Theorem. Given \( t \in [1, \infty] \) and \( \alpha, \beta \in \mathbb{R} \), there exists constant \( K \) such that for all functions \( f \in L^1(\mathbb{T}^d) \),

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^\alpha \|f|_\beta \|_r,
\]

for all finite \( E \subset \mathbb{Z}^d \) if and only if \( \alpha, \beta, t \) satisfy the following conditions (see Figure 1):

1. \( \beta < d/t' \) if \( 1 < t < \infty \), otherwise \( \beta \leq d/t' \) if \( t = 1 \);
2. \( \alpha > \max\{0, -1/t^*\} \) if \( \beta < 0 \) and \( 1 < t < 2 \), or \( \beta < d/t^* \) and \( t < 2 \);
3. \( d \alpha > \beta - d/t^* \) if \( \max\{0, d/t^*\} < \beta < d/t' \).

The shaded area of Figure 1(a) is the region of validity of the inequality for \( 1 < t < 2 \) with the boundary \( \beta = d/t' \) (\( \alpha > 1/2 \)) being included for \( t = 1 \). When \( 2 < t < \infty \) the region of validity is the shaded area of Figure 1(b).

2. Proof of sufficiency. Assuming the conditions of the theorem hold, let \( N \subset \mathbb{T}^d \), \( N' = \mathbb{T}^d - N \) be its complement, and \( f \) be a function in \( L^2(\mathbb{T}^d) \). We thus have \( f = f\chi_N + f\chi_{N'} \), where \( \chi_N \) is the characteristic function on \( N \), so that \( \hat{f} = (f\chi_N) + (f\chi_{N'}) \). Hence,

\[
\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq \left( \sum_{n \in E} |(f\chi_N)(n)|^2 \right)^{1/2} + \left( \sum_{n \in E} |(f\chi_{N'})(n)|^2 \right)^{1/2}
\]

by Minkowski's inequality; we shall estimate the two quantities separately. First, using Hölder's inequality,

\[
\left( \sum_{n \in E} |(f\chi_N)(n)|^2 \right)^{1/2} \leq \max\{|(f\chi_N)(n)| : n \in E\} |E|^{1/2} \leq \|f\chi_N\|_1 |E|^{1/2}
\]

\[
= \|f\chi_N|_x^\beta |x|^{-\beta} \|_1 |E|^{1/2} \leq \|f\chi_N|_x^\beta \|_1 \chi_N|_x^{-\beta} \|_r |E|^{1/2}.
\]
Considering the second term of (2.1), we have

\[(2.3) \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} \leq \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} = \|f \chi_N\|_2 \]

\[= \|f \chi_N \|_{L^2} \|x\|^{\beta} \|x\|^{-\beta} \|_{L^2} \|x\|^{-\beta} \|_{L^2} \]

where we used Hölder's inequality with \( r = t/2 \) and assumed \( t > 2 \).

If we assume that \( t \in [1, 2] \) we estimate the second term as follows:

\[(2.4) \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} = \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} \]

\[\leq \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}^d} 1^{2r'} \right)^{1/2r'} \]

\[= \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} \|E\|_{L^{1/r}} \]

\[\leq \left( \sum_{n \in \mathbb{Z}^d} \left| (f \chi_N \hat{\gamma}(n) \right|^2 \right)^{1/2} \|E\|_{L^{1/r}} \]

\[\leq \|f \chi_N \|_{L^2} \|E\|_{L^{1/r}} \]

where we used Hölder's inequality with \( r = t'/2 \), followed by Hausdorff-Young and a final application of Hölder's inequality.

Substituting (2.2)–(2.4) into (2.1), we obtain

\[(2.5) \left( \sum_{n \in \mathbb{E}} \left| \hat{f}(n) \right|^2 \right)^{1/2} \leq K\|f \|_{L^2} \|eta\|_{r'}, \]

where \( K = K_1 + K_2 \), with

\[K_1 = \|X_N \|_{L^2} \|E\|_{L^{1/r}}^{1/2}, \]

\[K_2 = \begin{cases} \|X_N \|_{L^2} \|E\|_{L^{1/r}}^{1/2} & \text{for } 1 \leq t \leq 2, \\ \|X_N \|_{L^\infty} \|E\|_{L^{1/r}}^{1/2} & \text{for } 2 < t \leq \infty. \end{cases} \]

Letting \( N = \{ x \in T^d : |x| < a \} \) for some \( a \), we now evaluate the above norms. First,

\[(2.6) \|X_N \|_{L^2} \|E\|_{L^{1/r}}^{1/2} = \int_{|x| \leq a} |x|^{-\beta r'} dx = W_d \int_0^a r^{-\beta r'} r^{d-1} dr \]

\[= (W_d/d - \beta r\}) a^{d-\beta r'}, \]
as $\beta < d/t'$, where $W_d = 2\pi^{d/2}/\Gamma(d/2)$. Second

$$\|X_N|\cdot|\cdot|^{-\beta}\|^d_{\ell^d} = \int_{x \leq |x| \leq \pi} |x|^{-\beta d} \, dx = W_d \int_a^{\pi} r^{-\beta d + d - 1} \, dr$$

$$= W_d (\beta t^d - d)^{-1} (a^{d-\beta d} - \pi^{d-\beta d}) \leq W_d (\beta t^d - d)^{-1} a^{d-\beta d},$$

as $\beta > \max\{0, d/t^d\}$. Thirdly if $\beta > 0$,

$$\|X_N|\cdot|\cdot|^{-\beta}\|_{\ell^\infty} = \sup\{|x|^{-\beta} : a \leq |x| \leq \pi\} = a^{-\beta}.$$

Now, letting $a = |E|^{-1/d}$ and substituting (2.6)–(2.8) into (2.5), we obtain

$$\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K|E|^\beta/d - 1/\beta \|f|_{\ell^\beta},$$

with $K = K_1 + K_2$, where $K_1 = (W_d/(d - \beta t'))^{1/t'}$ and

$$K_2 = \begin{cases} 1 & \text{for } 1 \leq t \leq 2, \\ (W_d/(\beta t^d - d))^{1/t} & \text{for } 2 < t \leq \infty, \end{cases}$$

provided $\max\{0, d/t^d\} < \beta < d/t'$.

Finally, if $\beta \leq 0$, $\|X_N|\cdot|\cdot|^{-\beta}\|_{\ell^\infty} = \pi$; hence, $K_2 = \pi|E|^{-1/\beta}$ and

$$K_1 = (W_d/(d - \beta t'))^{1/t'} a^{d-\beta} |E|^{1/t} = (W_d/(d - \beta t'))^{1/t'} |E|^{1/\beta}$$

upon letting $a = |E|^{-1/(d-\beta t')}$ for $t \neq 1$. If $t = 1$,

$$K_1 = \|X_N|\cdot|\cdot|^{-\beta}\|_{\ell^\infty} |E|^{1/2} = a^{-\beta} |E|^{1/2} = |E|^{-1/\beta}$$

upon letting $a = 1$. Thus

$$\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K|E|^{-1/\beta} \|f|_{\ell^\beta},$$

for $\beta \leq 0$ and $1 \leq t \leq 2$ with $K = K_1 + K_2$, where $K_2 = \pi$ and

$$K_1 = \begin{cases} 1 & \text{for } t = 1, \\ (W_d/(d - \beta t'))^{1/t'} & \text{for } 1 < t \leq 2. \end{cases}$$

We thus have the inequality holding along the line segment in the $(\alpha, \beta)$-plane given by $d\alpha = \beta - d/t^d$ for $\max\{0, d/t^d\} < \beta < d/t'$. When $t = 1$ we also have it holding at the endpoint $\beta = d/t' = 0$ since

$$\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq |E|^{1/2} \|f\|_{\ell^\infty} \leq |E|^{1/2} \|f\|_1.$$

These results may be extended as follows: If

$$\left( \sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K|E|^\alpha \|f|_{\ell^\beta},$$

holds for given $(\alpha, \beta, t)$, then it holds for $(\alpha', \beta', t')$ with $K$ replaced by $K \| |x|^{-\beta'}\|_\infty$, where $\alpha' \geq \alpha$, $\beta' \leq \beta$ and $t' > t$. To see this, notice that $|E|^\alpha' \geq |E|^\alpha$ and $\|f|_{\ell^\beta} \leq \| |x|^{-\beta'}\|_\infty \|f|_{\ell^\beta'}$. 

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This leaves only the case $\alpha = 0$ and $t \geq 2$; it can be established as follows:

$$
\bigg\| \hat{f} \bigg\|_2 = \bigg\| f \bigg\|_2 = \bigg\| f |x|^\beta |x|^{-\beta} \bigg\|_2 \leq \bigg\| f |x|^\beta \bigg\|_{t} \bigg\| x \bigg\|^{-\beta} \bigg\|_t
$$

with

$$
\bigg\| x \bigg\|^{-\beta} \bigg\|_t = W_d \int_0^\pi r^{-\beta t + d - 1} dr < \infty \quad \text{for } \beta \leq d/t^* (t > 2)
$$

and

$$
\bigg\| x \bigg\|^{-\beta} \bigg\|_t = \sup \{ |x|^{-\beta} : x \in T^d \} < \infty \quad \text{for } \beta \leq d/t^* = 0 \quad (t = 2).
$$

We have thus covered the stated region, and the sufficiency of the conditions is established.

**Remark.** In the interior of the region an alternate value of the constant is $K = \bigg\| |x|^{-\beta} \bigg\|_{2t/(t+2a,t-2)}$. To see this, notice that

$$
\left( \sum_{n \in E} | \hat{f}(n) |^2 \right)^{1/2} \leq |E|^{1/2r} \left( \sum_{n \in Z^d} | \hat{f}(n) |^{2r} \right)^{1/2r} \leq |E|^{1/2r} \bigg\| f |x|^\beta |x|^{-\beta} \bigg\|_{(2r)^r} \leq |E|^{1/2r} \bigg\| x^{-\beta} \bigg\|_{(2r)^r} \bigg\| f |x|^\beta \bigg\|_{(2r)^r} \bigg\|_{t}. 
$$

**3. Proof of necessity.** The necessity of the conditions is obtained by ruling out the remaining regions beginning with $\alpha \leq 0$. Firstly, we can not have $\alpha < 0$, since then $|E|^\alpha \to 0$ as $|E| \to \infty$. Secondly, considering the case $\alpha = 0$, $\beta = d/t^*$ with $t > 2$, the function $f \in L^1(T^d)$ defined by

$$
f(x) = \begin{cases} 
|x|^{-d/2} |\log |x||^{-1/2} & \text{for } |x| \leq 1/2, \\
0 & \text{otherwise}
\end{cases}
$$

provides a counterexample, since

$$
\bigg\| f \bigg\|_2 = W_d \int_0^{1/2} r^{-1} \log^{-1} r \, dr = \infty,
$$

while

$$
\bigg\| f |x|^{d/t^*} \bigg\|_t = W_d \int_0^{1/2} r^{-1} |\log r|^{-1/2} \, dr < \infty \quad \text{if } t > 2.
$$

This also rules out the region $\alpha = 0$, $\beta > d/t^*$ for $t > 2$. Thirdly, consider the region given by $\alpha = 0$, $\beta < d/t^*$ with $1 \leq t < 2$. Here define $f(x) = ||x|-1||^{-1/2}$ for $1/2 \leq |x| \leq 3/2$ and 0 otherwise, so that

$$
\bigg\| f \bigg\|_2 = W_d \int_{1/2}^{3/2} |r - 1|^{-1} r^{d-1} \, dr = \infty,
$$

while

$$
\bigg\| f |x|^\beta \bigg\|_t = W_d \int_{1/2}^{3/2} |r - 1|^{-1/2} r^{\beta t + d - 1} \, dr < \infty \quad \text{for } t < 2.
$$
Finally, the region given by $\alpha = 0$, $\beta > d/t^*$ for $t = 2$ is ruled out by $f(x) = |x|^{-d/2}$ since

$$
\|f\|_2^2 = W_d \int_0^1 r^{-1} dr = \infty ,
$$

while

$$
\|f|\beta\|_2 = W_d \int_0^1 r^{-1+2\beta} dr < \infty ,
$$

which completes the $\alpha \leq 0$ case.

Now consider the boundary $\beta > d/t^*$; here the function

$$f_\epsilon(x) = \begin{cases} |x|^{-d} \log|x|^{t-1} & \text{for } 0 < \epsilon \leq |x| \leq 1/2 , \\ 0 & \text{otherwise} \end{cases}
$$

provides the required counterexample as $\epsilon \to 0$. For assume that $E = \{0\}$; then

$$
\left( \sum_{n \in E} |f_\epsilon(n)|^2 \right)^{1/2} = |f_\epsilon(0)| = \left| W_d \int_\epsilon^{1/2} r^{-1} \log^{-1} r dr \right| \to \infty \quad \text{as } \epsilon \to 0 ,
$$

while

$$
\left\| f_\epsilon \right\|^\beta = W_d \int_\epsilon^{1/2} r^{\beta - dt + d - 1} \log r^{-t} r dr \leq \text{constant}
$$

for all $\epsilon > 0$ if $\beta t - dt + d - 1 \leq -1$ and $t > 1$ or $\beta t - dt + d - 1 > -1$ and $t = 1$. Hence, there is a contradiction for all $\alpha$ if $\beta \geq d/t^*$ and $t > 1$ or $\beta > d/t^*$ and $t = 1$.

Consider now the region $d\alpha < \beta - d/t^*$ for $\max\{0, d/t^*\} < \beta < d/t'$. Define

$$f_N(x) = \begin{cases} 1 & \text{if } x \in \Box_N , \\ 0 & \text{otherwise} \end{cases}
$$

where

$$\Box_N = \left\{ x = (x_1, \ldots, x_d) \in T^d: |x_1| \leq 1/N, \ldots, |x_d| \leq 1/N \right\}
$$

and $N$ is a positive integer. Now

$$f_N(n) = f_N(n_1, \ldots, n_d) = \int_{\Box_N} e^{-i n_1 x_1} \cdots e^{-i n_d x_d} dx_1 \cdots dx_d = 2^d n_1^{-1} \sin n_1/N \cdots n_d^{-1} \sin n_d/N .
$$

Thus, letting $E = \{ n = (n_1, \ldots, n_d) \in Z^d: |n_1| \leq N, \ldots, |n_d| \leq N \}$, we have

$$
\left( \sum_{n \in E} |f_N(n)|^2 \right)^{1/2} = 2^d \left( \sum_{|n_1| \leq N} \frac{\sin^2 n_1/N}{n_1^2} \cdots \sum_{|n_d| \leq N} \frac{\sin^2 n_d/N}{n_d^2} \right)^{1/2} \sim 1/\sqrt{N^d} \quad \text{as } N \to \infty
$$

since

$$
\lim_{N \to \infty} \sum_{|n| \leq N} \left( \frac{n}{N} \right)^2 \sin^2 \left( \frac{n}{N} \right) = \int_{-1}^{1} x^{-2} \sin^2 x dx ,
$$
while

\[ |E|^\alpha \|f_N|\beta \|_t = N^{d\alpha} \left( \int_{\Delta_N} |x|^\beta t \, dx \right)^{1/t} \]

\[ \leq N^{d\alpha} \left( \int_{|x| \leq d^{1/2}/N} |x|^\beta t \, dx \right)^{1/t} \]

\[ = \left( W_d/(\beta t + d) \right)^{1/t} d^{(\beta/2)+(d/2t)} N^{d\alpha - \beta - d/t}, \]

assuming \( \beta > d/t^* > -d/t \). We thus have a contradiction if \(-d/2 > d/\alpha - \beta - d/t\), that is, if \(d\alpha < \beta - d/t^*\), as required.

Finally, define \( g_N(x) = f_N(x - 1) \), \( f_N \) as above, and \( 1 = (1, \ldots, 1) \), for \( N > 1 \) being an integer. With \( E \) as above we have

\[ \left( \sum_{n \in E} |\hat{g}_N(n)|^2 \right)^{1/2} \sim N^{-d/2} \quad \text{as } N \to \infty \]

since \( |\hat{g}_N(n)| = |\hat{f}_N(n)| \). Therefore consider, for \( \beta < 0 \),

\[ |E|^\alpha \|g_N|\beta \|_t = N^{d\alpha} \left( \int_{\Delta_N} |y + 1|^\beta t \, dy \right)^{1/t} = N^{d\alpha} \left( \int_{\Delta_N} |y|^\beta t \, dy \right)^{1/t} \]

\[ \leq \left( \frac{1}{2} \right)^{\beta} N^{d\alpha} \left( \int_{\Delta_N} dy \right)^{1/t} = 2^{d/t - \beta} N^{d\alpha} N^{-d/t}. \]

We thus obtain a contradiction as \( N \to \infty \) if \(-d/2 > d/\alpha - d/t\), that is, \( \alpha < -1/t^* \) for \( \beta < 0 \).

All the required regions are now eliminated and necessity is established which completes the proof of the Theorem.

REFERENCES


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