CONTINUOUS FUNCTIONS ON POLAR SETS

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Abstract. Let $\Omega$ be a second countable Brelot harmonic space with a positive potential. If $K$ is a compact subset of $\Omega$ with more than one point, then $K$ is a polar set iff every positive continuous function on $K$ can be extended to a continuous potential on $\Omega$. This is a generalization of the result proved by H. Wallin for the special case $\Omega = \mathbb{R}^n$ $(n \geq 3)$ with Laplace harmonic space.

The objective of this paper is to generalize to axiomatic spaces of harmonic functions a well-known result of H. Wallin [6] which states that a compact set $K \subset \mathbb{R}^n$, $n \geq 3$, has Newtonian capacity 0 iff every positive continuous function on $K$ is the restriction to $K$ of a positive Newtonian potential on the whole space. We give the generalization of this result at the setting of Brelot space [3], which contains a positive potential, and the topology of the space has countable space for open sets. Accordingly, we prove

**Theorem 1.** Let $K$ be a compact subset contained in $\Omega$, a Brelot space as above, such that every positive continuous function on $K$ can be uniformly approximated on $K$ by positive superharmonic functions on $\Omega$. Further, let $K$ contain at least two points. Then $K$ is polar.

**Theorem 2.** Let $K$ be a compact polar subset of $\Omega$, $f_0$ a positive continuous function on $K$, and $F_0$ a relatively compact open neighborhood of $K$. Then there is a continuous potential $p$ on $\Omega$ such that $p = f_0$ on $K$ and $p$ is harmonic on $\Omega \setminus F_0$.

In view of the fact that there are Brelot spaces of the nature mentioned here in which point sets need not be polar [2], the restriction in Theorem 1 is quite warranted. We can write $K = F \cup G$, with $F$ and $G$ closed subsets of $\Omega$, neither of which is a subset of the other. Let $z$ be in $G$ and not in $F$. Then, by the Tietze extension theorem, for each positive integer $n$ there is a continuous function $f_n$ on $K$ such that $f_n > 0$ on $K$, and $f_n(x) = 1$ for all $x$ in $F$ and $f_n(z) = 1/2^n$. Choose $\varepsilon$ such that $0 < \varepsilon < 1/2$. Then, by hypothesis, there is a positive superharmonic function $q_n$ on $\Omega$ such that $|f_n(x) - q_n(x)| < \varepsilon/2^{n-1}$, i.e.,

$$1 - \frac{\varepsilon}{2^{n-1}} < q_n(x) < 1 + \frac{\varepsilon}{2^{n-1}}$$

for all $x$ in $F$.
and
\[ \frac{1}{2^n} - \frac{\epsilon}{2^{n-1}} < q_n(z) < \frac{1}{2^n} + \frac{\epsilon}{2^{n-1}}. \]

Put \( q(x) = \sum_{n=1}^{\infty} q_n(x) \) on \( \Omega \). Then, obviously, \( q \) is a superharmonic function and \( q = \infty \) on \( F \). Hence, \( F \) is a polar set. Similarly, \( G \) is also a polar set. Thus, \( K \) is a polar set.

The proof of Theorem 2 relies on first providing an approximation which is in line with the conclusion of the Theorem 1. We also include the following application, which is a generalization of another result of H. Wallin (see [6, Remark 4, p. 62]).

**Theorem 3.** Let \( \Omega \) be a selfadjoint Brelot space [5] with second axiom of countability, \( \delta \) a regular domain in \( \Omega \), and \( K \) a compact polar set \( \subset \delta \). Assume that the constant function 1 is superharmonic on \( \Omega \). Then, given any positive continuous function \( f \) on \( K \), there is a continuous potential \( p \) on \( \Omega \) such that \( p \) is harmonic on \( \delta \), \( f = p \) on \( K \), and \( D_p(\delta) < \infty \), where \( D_p \) is the gradient measure, in the sense of Maeda, associated with \( p \) [5].

**Proof.** By Theorem 2 there is a continuous potential \( q_1 \) on \( \Omega \) such that \( f = q_1 \) on \( K \). Since \( f \) is continuous on \( K \), there is a constant \( b \) such that \( 0 < f < b \) on \( K \). Put \( q = \min(q_1, b) \) on \( \Omega \). Then \( q \) is a bounded continuous potential on \( \Omega \) and \( q = f \) on \( K \). Let

\[
p(x) = \begin{cases} \int q(z) \, d\rho_{\delta}^p(z) & \text{if } x \in \delta, \\ q(x) & \text{if } x \in \Omega \setminus \delta. \end{cases}
\]

Then \( p \) is a bounded continuous potential on \( \Omega \), and \( p \) is harmonic on \( \delta \). By a result of Maeda [5], \( D_p(\delta) < \infty \). Further, since \( K \subset \delta \), \( f(x) = q(x) = p(x) \) for every \( x \) in \( K \). The proof is complete.

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**1. Preliminaries.** From now on, let \( K \) denote a compact polar set in a Brelot space \( \Omega \). The next theorem is a converse of Theorem 1.

**Theorem 4.** Given a positive continuous function on \( K \) and an \( \epsilon > 0 \), there exists a continuous potential \( p \) on \( \Omega \) such that \( |f - p| < \epsilon \) on \( K \).

**Proof.** Let \( \{U_n\}_{n=1}^{\infty} \) be a decreasing sequence of relatively compact open subsets of \( \Omega \) such that \( U_n \supset U_{n+1} \supset U_{n+1} \) for \( n = 1, 2, 3, \ldots \), and \( \cap_{n=1}^{\infty} U_n = K \). For each \( n \), let \( f_n \) be a continuous extension of \( f \) to \( \Omega \) such that \( f_n \geq 0 \) on \( \Omega \), and \( f_n \) has support contained in \( U_n \). Taking the infimum at each stage, we may assume that \( \{f_n\}_{n=1}^{\infty} \) is a decreasing sequence. Note that \( f_n \rightarrow 0 \) outside \( K \) and \( f_n \rightarrow f \) on \( K \). From the fact that \( \{R_f^K\}_{n=1}^{\infty} = \{R_{f_n}^K\}_{n=1}^{\infty} \) is a decreasing sequence, it can be proved that \( R_{f_n} \rightarrow R_f^K \) pointwise on \( \Omega \). However, it is routine to prove that \( R_{f_n}^K = f \) on \( K \). Hence, \( R_{f_n} \rightarrow f \) on
Since, for each \( n \), \( Rf_n \) is a continuous potential on \( \Omega [2] \), by Dini's theorem, the convergence is uniform on \( K \). Thus, there is an \( m \) such that \(|Rf_n - f| < \epsilon\) on \( K \) if \( n \geq m \).

**Theorem 5.** Let \( f_0 \) be a positive continuous function on \( K \), and let \( F_0 \) be a relatively compact open neighbourhood of \( K \). Put \( F = \overline{F_0} \) and let \( f \) be a nonnegative continuous extension of \( f_0 \) to \( \Omega \), such that \( f > 0 \) on \( F \). Then, given \( \epsilon > 0 \), there is a continuous potential \( p \) on \( \Omega \) such that \( p < f \) on \( F \) and \( p \geq f_0 - \epsilon \) on \( K \). In fact, \( p \) can be chosen to be harmonic outside \( F \).

**Proof.** We can assume \( \epsilon > 0 \) is small enough so that \( f_0 - \epsilon > 0 \) on \( K \). Apply the proof of the previous theorem to the function \( f_0 - \epsilon \). Let \( U_n \) be a sequence of relatively compact open subsets of \( \Omega \) such that \( U_n \supseteq \overline{U}_{n+1} \supseteq U_{n+1}, n = 1, 2, 3, \ldots \). Let \( g_n \) be a nonnegative, continuous extension of \( f_0 - \epsilon \) to \( \Omega \), with the support of \( g_n \subseteq U_n \), and such that \( Rg_n \) converges to \((f_0 - \epsilon)\chi_K \) on \( \Omega \) as \( n \to \infty \).

Now, let \( \eta > 0 \) such that \( \eta < \epsilon \) and \( f_0 - \epsilon + \eta < f_0 \) on \( K \). Since \( Rg_n \) converges to \( f_0 - \epsilon \) uniformly on \( K \), there is an integer \( n_1 \) such that \( Rg_n < f_0 - \epsilon + \eta \) on \( K \) if \( n \geq n_1 \). Since \( Rg_n \) and \( f - \epsilon + \eta \) are two continuous functions on \( \Omega \), the set \( V \), defined as \( \{ x \in \Omega : Rg_n(x) < (f(x) - \epsilon + \eta) \} \), is an open set containing \( K \). Further, \( Rg_n \) is a decreasing sequence of functions on \( \Omega \). Hence,

\[
Rg_n(x) < f(x) - \epsilon + \eta \quad \text{for every } x \text{ in } V \text{ and every } n \geq n_1.
\]

Now \( F_0 \cap V \) is an open set which clearly contains \( K \). This implies that there is an integer \( n_2 \) such that

\[
\overline{U}_n \subseteq V \cap F \quad \text{if } n \geq n_2.
\]

Put \( n_3 = \max(n_1, n_2) \). Choose \( \alpha > 0 \) such that \( f \geq \alpha \) on \( F \). Observe that \( F \setminus U_{n_3} \) is a compact subset of \( \Omega \), and \( Rg_n \) decreases to zero on \( F \setminus U_{n_3} \) as \( n \to \infty \). Hence, by Dini's theorem, the convergence is uniform, i.e., there is an integer \( n_4 \) such that \( Rg_n \leq \alpha/2 \) on \( F \setminus U_n \) if \( n \geq n_4 \).

Put \( m = \max(n_3, n_4) \) and \( p = Rg_m \). Then it follows that \( p \) is a continuous potential; on \( F \setminus U_{n_3} \),

\[
p = Rg_m \leq Rg_{n_4} \leq \alpha/2 < f;
\]

and, lastly, using (1) and (2),

\[
p = Rg_m \leq Rg_{n_3} \leq Rg_{n_1} \leq f - \epsilon + \eta < f \quad \text{on } U_{n_3}.
\]

Thus \( p < f \) on \( F \). Further, on \( K, f_0 - \epsilon = g_m \leq Rg_m = p \). The proof is complete.

**2. Proof of Theorem 2.** Now we prove the main theorem.

Put \( F = \overline{F_0} \). Choose \( f \), a nonnegative continuous extension of \( f_0 \) to \( \Omega \), such that \( f > 0 \) on \( F \) and \( f \) is zero outside a compact subset of \( \Omega \). Let \( \{U_n\} \) be a sequence of relatively compact open subsets of \( \Omega \) such that \( U_{n+1} \subseteq \overline{U}_{n+1} \subseteq U_n \) for \( n = 0, 1, 2, \ldots \), and \( K = \bigcap_{n=0}^{\infty} U_n \).

We can assume that \( U_0 \subseteq F_0 \). Fix a small \( \epsilon > 0 \).

By the previous theorem there is a continuous potential \( p_0 \) on \( \Omega \) such that:

(i) \( p_0(x) < f(x) \) for every \( x \) in \( F \),

(ii) \( p_0(x) \geq f_0(x) - \epsilon \) for every \( x \) in \( K \), and

(iii) \( p_0 \)
is a harmonic function on the open set $\Omega \setminus \overline{U}_0$. Let $f_1 = \max(f - p_0, 0)$ on $\Omega$. Then $f_1$ is a continuous function on $\Omega$ and $f_1 > 0$ on $F$. Applying the previous theorem to $f_1$ and then proceeding inductively, we obtain a sequence $\{p_n\}_{n=1}^\infty$ of continuous potentials on $\Omega$ such that: (i) $p_i$ is a harmonic function on the open set $\Omega \setminus \overline{U}_i$ for every $i$, (ii) $\sum_{i=0}^n p_i < f$ on $F$ for every $n$, and (iii) $\sum_{i=0}^n p_i \geq f_n - \epsilon/2^n$ on $K$ for every $n$.

Put $p = \sum_{i=0}^\infty p_i$. It is clear that $p$ is a potential function on $\Omega$. Further, on $K$,

$$f_0 - \epsilon/2^n \leq \sum_{i=0}^n p_i < f = f_0$$

for every $n$.

Taking the limit as $n \to \infty$, we conclude that $p = f_0$ on $K$.

Now $p_i$ is a harmonic function outside $\overline{U}_i \subset F_0 \subset F$ for every $i$. Hence, $\sum_{i=0}^n p_i$ is a harmonic function outside $F$. Therefore, on every connected component of $\Omega \setminus F$, $p = \lim_{n \to \infty} \sum_{i=0}^n p_i$, is either a harmonic function or identically $+\infty$. But $p$ is a superharmonic function on $\Omega$. This implies that $p$ is finite on a dense subset of $\Omega$, which eliminates the second possibility. Hence, $p$ is a harmonic function on $\Omega \setminus F$.

Thus, we are left with proving the continuity of $p$ on $\Omega$.

Let $z$ be in $\Omega \setminus F$. Since $p$ is a harmonic function on the open set $\Omega \setminus F$, it is continuous at $z$. Let $z$ be in $K$. Then

$$p(z) \leq \liminf_{x \to z} p(x) \quad \text{(by lower semicontinuity of } p)$$
$$\leq \limsup_{x \to z} p(x)$$
$$= \limsup_{x \to z} p(z) \quad \text{(as } F_0 \text{ is an open neighbourhood of } z)$$
$$\leq f(z) \quad \text{(as } f \text{ is continuous at } z)$$
$$= f_0(z)$$
$$= p(z) \quad \text{(as } p = f_0 \text{ on } K).$$

Hence, $p$ is continuous at $z$.

Now, let $z$ in $F \setminus K$. Then there exists an integer $N$ such that $z \notin U_n$ if $n \geq N$. By writing $p(x) = \sum_{i=0}^{N-1} p_i(x) + \sum_{i=N}^\infty p_i(x)$, we see that $\sum_{i=N}^\infty p_i(x)$ is a harmonic function in an open neighbourhood of $z$. In particular, $\sum_{i=N}^\infty p_i(x)$ is continuous at $z$. Being a finite sum of continuous functions at $z$, $\sum_{i=0}^{N-1} p_i$ is a continuous function at $z$. Hence, $p$ is continuous at $z$. Thus, $p$ is continuous on $\Omega$, and the proof is complete.

**Corollary 6.** Assume that the constant functions are harmonic on $\Omega$. If $K$ is a compact polar subset of $\Omega$, then every continuous function on $K$ can be extended to a superharmonic function on $\Omega$.

**Bibliography**


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