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SHORTER NOTES

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A SHORT PROOF OF THE BIRKHOFF-SMALE THEOREM

T. MROWKA

ABSTRACT. A short proof of the Birkhoff-Smale theorem on homoclinic points of diffeomorphisms is given. The proof is elementary in that only the local properties of a hyperbolic fixed point are assumed.

The following theorem is fundamental in understanding how homoclinic points complicate the dynamics of a diffeomorphism.

THEOREM (BIRKHOFF AND SMALE). *Let $f\colon M \to M$ be a diffeomorphism. If f has a transverse homoclinic point q, then f has a periodic point in any neighborhood of q.*

This theorem was proved by Birkhoff [1] for diffeomorphisms of the plane and then generalized by Smale [2]. But Smale's proof involves a long argument in which he constructs a Bernoulli shift on an invariant Cantor set. The proof given here uses only the local theory for a hyperbolic fixed point of a diffeomorphism.

We recall some definitions. A fixed point p is called hyperbolic if $T_pf\colon T_pM \to T_pM$ has no eigenvalues of modulus one. The stable manifold $W^{\mathrm{s}}(p)$ of a hyperbolic fixed point p is the set of $x \in M$ such that $f^n(x) \to p$ as $n \to \infty$. The unstable manifold $W^{\mathrm{u}}(p)$ is the stable manifold of f^{-1} at p. A point $q \neq p$, where $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$ intersects transversely, is called a transverse homoclinic point.

PROOF. Let p and q be as above. $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$ intersect transversely at p, so we can find U, a neighborhood of p, and a diffeomorphism $\phi\colon B^{\mathrm{u}} \times B^{\mathrm{s}} \to U$, where $B^{\mathrm{u}} \subset W^{\mathrm{u}}(p)$ and $B^{\mathrm{s}} \subset W^{\mathrm{s}}(p)$ are open balls about p so that $\phi(B^{\mathrm{u}} \times \{p\}) = B^{\mathrm{u}}$ and $\phi(\{p\} \times B^{\mathrm{s}}) = B^{\mathrm{s}}$. Let $\pi_1\colon U \to B^{\mathrm{u}}$ and $\pi_2\colon U \to B^{\mathrm{s}}$ be the induced projections. The λ-lemma tells us that any disk transverse to the stable (resp. unstable) manifold becomes arbitrarily C^1 close to some disk in the unstable (stable) manifold under forward (backward) iteration (see Palis [4]). Thus for any neighborhood V of q we

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can find $D^{\mathrm{u}} \subset B^{\mathrm{u}}$ and $D^{\mathrm{s}} \subset B^{\mathrm{s}}$, closed neighborhoods of p and positive integers m and n with the following properties:

(i) Let $N = \phi(D^{\mathrm{u}} \times D^{\mathrm{s}})$; then C, the connected component of $f^m(N) \cap f^{-n}(N)$ containing q, is contained in V.

(ii) For any $(u, s) \in D^{\mathrm{u}} \times D^{\mathrm{s}}$, $f^m(\phi(D^{\mathrm{u}} \times \{s\})) \cap f^{-n}(\phi(\{u\} \times D^{\mathrm{s}})) \cap C$ consists of exactly one point, and $f^m(\phi(D^{\mathrm{u}} \times \{s\}))$ and $f^{-n}(\phi(\{u\} \times D^{\mathrm{s}}))$ intersects transversely at this point.

The latter condition ensures that the maps $g_1\colon f^{-m}(C) \to N$ and $g_2\colon f^n(C) \to N$ given by

$$g_1(\phi(u, s)) = \phi(\pi_1(f^{m+n}(\phi(u, s))), s),$$
$$g_2(\phi(u, s)) = \phi(u, \pi_2(f^{-m-n}(\phi(u, s))))$$

are bijections. The g_i's have compact domain and Hausdorff range; hence, they are homeomorphisms. Define $h\colon N \to N$ by

$$h(\phi(u, s)) = \phi\left(\pi_1\left(g_1^{-1}(\phi(u, s))\right), \pi_2\left(g_2^{-1}(\phi(u, s))\right)\right).$$

The Brouwer fixed point theorem gives a fixed point x_0 of h. Clearly x_0 is also a fixed point of g_1 and g_2. Finally, note that $f^{m+n} = g_2^{-1} \circ g_1$, so x_0 is a periodic point of f and $f^m(x_0) \in V$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720