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CHARACTERIZATIONS OF NOETHERIAN AND HEREDITARY RINGS

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ABSTRACT. We characterize the left Noetherian rings by the existence of decompositions of left modules into direct sums of an injective submodule and a submodule containing no injective submodule except 0. We also prove that a left Noetherian ring is left hereditary iff the suspension of each left ideal (see [7]) is injective or, equivalently, the above decomposition is unique.

0. Let A be an Abelian group. Then $A = D \oplus B$, where D is divisible and B is reduced in the sense of Kaplansky [**8**]; that is, B has no divisible submodule except 0. If this holds for left modules over a unitary ring R, with "injective" and "i-reduced" replacing "divisible" and "reduced", respectively, then R must be left Noetherian. The proof requires the theorem of Eilenberg-Bass (see, e.g., Theorem 20.1, p. 111, of [**2**]) which characterizes left Noetherian by the requirement that direct sums of injective left modules are injective. Furthermore, if the decomposition, which will be referred to as an *i-decomposition*, is *unique* in the sense that D is the unique injective submodule of A for which $A = D \oplus B$, with B i-reduced, then every sum of injective submodules is injective, and hence R must be left hereditary by a theorem of Matlis [**9**]. At the same time, it will be shown that a left Noetherian ring is left hereditary iff the suspension of each left ideal is injective.

1. Throughout this paper, R denotes a unitary ring.

THEOREM 1. *The following conditions are equivalent*:

(i) *R is left Noetherian.*

(ii) *The direct sum of any countably infinite family of injective left R-modules is injective.*

(iii) *Any left R-module admits an i-decomposition.*

(iv) *Every injective left R-module is a direct sum of indecomposable left R-modules.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) is known, and due to Eilenberg-Bass. The implication (i) $\Rightarrow$ (iii) is immediate; and (i) $\Leftrightarrow$ (iv) is the theorem of Matlis [**9**] and Papp [**10**]. Let us prove (iii) $\Rightarrow$ (ii). Let $\{S_n;\ n = 1, 2, 3, \dots\}$ be a countably infinite

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family of injective left R-modules. Let D and B realize an i-decomposition of $S = \oplus_{n=1}^{\infty} S_n$. For each positive integer k, $T_k = \oplus_{n=1}^{k} S_n$ is an injective submodule of S. B being i-reduced, we must have $E(T_k \cap D) = T_k$ by which we deduce that $T_k \subseteq D$, where $E(T_k \cap D)$ denotes the injective hull (included in T_k) of $T_k \cap D$. In this way, $S = D$, which is injective. Q.E.D.

2. For convenience, we will "borrow" some notation from the homotopy theory of modules introduced by B. Eckmann and P. Hilton (for more details, see [**7**, Chapter 13; **6** and **5**]). For any left R-module A, we define the "*suspension*" of A to be $S(A) = E(A)/A$, where $E(A)$ denotes the injective hull of A. Let $f\colon A \to B$ be a morphism of left R-modules; we denote by $f \simeq_i 0$ the fact that f can be extended to any module containing A. We write $\bar{\pi}(A, B) = 0$ if, for any $f\colon A \to B$, $f \simeq_i 0$. We say that a left R-module B is *FG-injective* if $\bar{\pi}(A, B) = 0$ for any finitely generated left R-module A. The following properties hold:

(1) $f \simeq_i 0$ iff f can be extended to the injective hull $E(A)$ of A or, equivalently, to any injective extension E of A.

(2) If $f = g + h$, then $g \simeq_i 0$ and $h \simeq_i 0$ imply $f \simeq_i 0$.

(3) If $f = \alpha \circ \beta$, then either $\alpha \simeq_i 0$ or $\beta \simeq_i 0$ implies $f \simeq_i 0$.

(4) For any injective extension E of A, $S(A)$ is injective (or FG-injective, or FP-injective) iff E/A is.

(5) If $\bar{\pi}(I, B) = 0$ for each left ideal I of R, then B is injective.

(6) B is FP-injective iff $\bar{\pi}(A, B) = 0$ for any finitely presented left R-module A.

LEMMA. *Assume that R satisfies*:

($*$) *For any left ideal I of R, $S(I)$ is FG-injective.*

Then the sum of any FG-injective submodules of a left R-module is FG-injective and hence FP-injective.

PROOF. Let $\{V_i;\ i \in J\}$ be a family of FG-injective submodules of a left R-module; we need to show that $\sum_{i \in J} V_i$ is FG-injective. We may reduce the problem to the case when J is finite, and so it suffices to consider $J = \{1, 2\}$. For any positive integer m, let $\mathfrak{F}^m$ be the family of all R-modules which can be spanned by m elements. We will prove the following by induction on m:

$$(**) \qquad \bar{\pi}(F, V_1 + V_2) = 0 \quad \text{for any } F \in \mathfrak{F}^m.$$

Firstly, for $m = 1$, let $F = Ra \in \mathfrak{F}^1$ and let $f\colon F \to V_1 + V_2$. $f(a) = v_1 + v_2$ for some $v_i \in V_i$, $i = 1, 2$. Since V_i is FG-injective, there exists $\alpha_i\colon E(R) \to V_i$ such that $\alpha_i(r) = rv_i$ for all $r \in R$. Let $I = \{r \in R;\ r(v_1 + v_2) = 0\}$; and set $S = E(R)/I$ which is FG-injective by ($*$). Define $\beta\colon F \to S$ and $\alpha\colon S \to V_1 + V_2$ by $\beta(ra) = [r] \in S = E(R)/I$ and $\alpha([\bar{r}]) = \alpha_1(\bar{r}) + \alpha_2(\bar{r})$ for $r \in R$ and $\bar{r} \in E(R)$. We get $f = \alpha \circ \beta$. Now $\beta \simeq_i 0$, because S is FG-injective. So $f \simeq_i 0$, and thus ($**$) holds for $m = 1$.

Next, we prove ($**$) for $m > 1$, assuming that ($**$) is true if m is replaced by $m - 1$. Let $F = Ra_1 + Ra_2 + \cdots + Ra_m \in \mathfrak{F}^m$. We need show that $f \simeq_i 0$ for $f\colon F \to V_1 + V_2$. Consider $f_1 = f/Ra_1$. By the inductive hypothesis, $f_1 \simeq_i 0$, i.e. there is $f_1'\colon E(F) \to V_1 + V_2$ such that $f_1'/Ra_1 = f_1$. Let $g = f_1'/F$ and let $h = f - g$. Then

$h(a_1) = 0$, and so there exists α: $H = F/(Ra_1) \to V_1 + V_2$ such that $h = \alpha \circ p$, where p: $F \to H = F/(Ra_1)$ is the canonical projection. Clearly, $H \in \mathfrak{F}^m$; again by the inductive hypothesis, $\alpha \simeq_i 0$, whence $h \simeq_i 0$. But obviously, $g \simeq_i 0$; thus $f = g + h \simeq_i 0$. Q.E.D.

As a consequence, if R is a left Noetherian ring verifying $(*)$, then $\Sigma_{i \in J} V_i$ is injective when V_i $(i \in J)$ all are.

We also observe that if $\{V_i;\ i \in J\}$ is a family of injective submodules of a left module over a left Noetherian ring, and if $\Sigma_{i \in J} V_i$ admits a unique i-decomposition, then $\Sigma_{i \in J} V_i$ is injective. At this stage, using a theorem of Matlis [**9**] (see, e.g., Proposition 10 of [**3**]), we may conclude

THEOREM 2. *The following conditions are equivalent*:

(i) *R is left Noetherian and left hereditary.*

(ii) *R is left Noetherian and satisfies* $(*)$.

(iii) *Any left R-module admits a unique i-decomposition.*

(iv) *Every sum of injective submodules of a left R-module is injective.*

The equivalence (i) $\Leftrightarrow$ (ii) includes Corollary 14.B of [**4**].

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