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ASYMPTOTICS FOR SOLUTIONS OF SMOOTH RECURRENCE EQUATIONS

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ABSTRACT. It is shown that convergent solutions of a smooth recurrence equation whose gradient satisfies a certain "nonunimodularity" condition can be approximated by an asymptotic expansion. The lemma used to show this has some features in common with Poincaré's theorem on homogeneous linear recurrence equations. An application to the study of polynomials orthogonal with respect to the weight function $\exp(-x^6/6)$ is given.

1. Introduction. The aim of these notes is to prove the following

THEOREM. *Let $k \geqslant 0$ and $m \geqslant 1$ be integers, and let H be a complex-valued function of $k+2$ real variables $x_0, \ldots, x_{k+1}$, all of whose partial derivatives of order $\leqslant m$ are continuous in a neighborhood of the origin* $\mathbf{o}$. *Assume that*

$$\sum_{j=0}^{k} z^j \partial_j H(\mathbf{o}) \neq 0 \tag{1}$$

holds for all complex numbers z with $|z| = 1$ (∂_j abbreviates $\partial/\partial x_j$). Let the numbers y_n with

$$\lim_{n \to \infty} y_n = 0 \tag{2}$$

form a solution of the recurrence equation

$$H(y_n, y_{n+1}, \ldots, y_{n+k}, 1/n) = 0 \qquad (n \geqslant 1). \tag{3}$$

Then there are numbers $c_1, \ldots, c_m$ such that

$$y_n = \sum_{l=1}^{m} c_l n^{-l} + o(n^{-m}) \tag{4}$$

as $n \to \infty$. Moreover, c_l for $1 \leqslant l \leqslant m$ depends only on the ith partial derivatives of H for $1 \leqslant i \leqslant l$.

At the end of the paper, we will give an application of this result to show the existence of an asymptotic expansion for the ratio of the leading coefficients of

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consecutive polynomials orthonormal with respect to the weight function $\exp(-x^{-6}/6)$. (See [**2**, Chapter 11, p. 351 ff.] for a discussion of asymptotic expansions.) (Added in proof. Further applications are given in [**9**].) The proof of the above Theorem depends on the following Lemma, which we hope will be interesting in its own right. It should be compared to a theorem of H. Poincaré [**8**] on homogeneous recurrence equations (see also [**4**, Chapter XVII, p. 526 and **7**, Kapitel 10, §6, p. 300]).

LEMMA. *Let f and g be complex-valued functions defined on positive integers such that*

$$\sum_{j=0}^{k} \lambda_{nj} f(n+j) = g(n) \qquad (n > 0), \tag{5}$$

where λ_{nj} are complex numbers for which

$$\lambda_{nj} \to \lambda_j \quad \text{as } n \to \infty, \tag{6}$$

and such that the "characteristic polynomial"

$$P(z) = \sum_{j=0}^{k} \lambda_j z^j \tag{7}$$

has no roots of absolute value 1. *Let $\alpha < 0$, and suppose that f is bounded and*

$$\lim_{n\to\infty} g(n)/n^{\alpha} = 0. \tag{8}$$

Then

$$\lim_{n\to\infty} f(n)/n^{\alpha} = 0 \tag{9}$$

as well.

2. Proofs. Before we turn to the proof of the Lemma, observe that the assumption about the polynomial in (7) having no roots of absolute value 1 cannot be dropped, as shown by the equation

$$f(n+1) - f(n) = 1/n^2 \qquad (n > 0),$$

a solution of which is

$$f(n) = -\sum_{l=n}^{\infty} \frac{1}{l^2}.$$

PROOF OF THE LEMMA. In virtue of the assumption about the roots of $P(z)$, $1/P(z)$ has a Laurent expansion

$$\frac{1}{P(z)} = \sum_{l=-\infty}^{\infty} a_l z^l \tag{10}$$

absolutely convergent in an annulus $\rho^{-1} < |z| < \rho$ $(\rho > 1)$. The relation $P(z)(1/P(z)) = 1$ and the uniqueness of the Laurent expansion implies that

$$\sum_{j=0}^{k} \lambda_j a_{l-j} = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases} \tag{11}$$

Put $\lambda_{nj} = \lambda_j$ for $n \leqslant 0$, and extend f and g to arguments $n \leqslant 0$ as follows. Put $f(n) = 0$ for all $n \leqslant 0$ and then determine $g(n)$ for $n \leqslant 0$ from (5). Clearly, we will have $g(n) = 0$ for all but finitely many $n \leqslant 0$, and (5) will be valid for all n with $-\infty < n < \infty$. Taking a large positive n, (5) implies

$$\sum_{j=0}^{k} \lambda_j f(n+l+j) = g(n+l) + \sum_{j=0}^{k} (\lambda_j - \lambda_{n+l,j}) f(n+l+j)$$

for every l. Multiplying this equation by a_l and summing for $-\infty < l < \infty$, the series obtained on both sides will converge absolutely in view of the boundedness of f and g. Taking (11) into account, we obtain

$$f(n) = \sum_{l=-\infty}^{\infty} a_l g(n+l) + \sum_{l=-\infty}^{\infty} \sum_{j=0}^{k} a_l (\lambda_j - \lambda_{n+l,j}) f(n+l+j).$$

The first sum here is $o(n^\alpha)$ by (8), and the contribution of the second sum with $|l| \geqslant n/2$ is

$$O((\rho+1)/2)^{-n/2} = o(n^\alpha)$$

in view of the boundedness of f and that of $\lambda_j - \lambda_{n+l,j}$ (cf. (6)), and by virtue of the convergence of (10) for $\rho^{-1} < |z| < \rho$. Therefore, for every $\varepsilon > 0$ there is an n_ε such that

$$|f(n)| < \varepsilon n^\alpha + \varepsilon \max\{|f(n+l)|: -n/2 < l < n/2 + k\} \tag{12}$$

holds for all $n \geqslant n_\varepsilon$. Indeed, to account for the second term on the right-hand side, it is enough to make sure that

$$\sum_{l:\,|l|<n/2} \sum_{j=0}^{k} |a_l(\lambda_j - \lambda_{n+l,j})| < \varepsilon,$$

and this will indeed hold for large enough n in view of (6) and the absolute convergence of (10) for $z = 1$.

Writing $F(x) = \sup\{|f(l)|: l \geqslant x\}$ for any real x, which makes sense as f is bounded, we can conclude from (12) that $F(x) < \varepsilon x^\alpha + \varepsilon F(x/2)$ holds for all real $x \geqslant n_\varepsilon$ (note that $\alpha < 0$). Using this repeatedly, with $x/2^l$ replacing x for $0 \leqslant l \leqslant q$, where q is the largest integer $\leqslant \log_2(x/n_\varepsilon)$, we obtain that

$$F(x) < \sum_{l=0}^{\infty} \varepsilon^{l+1} \left(\frac{x}{2^l}\right)^\alpha + \varepsilon^{q+1} F\left(\frac{x}{2^{q+1}}\right).$$

Noting that $F(x/2^{q+1}) \leqslant F(0)$ and $\varepsilon^{q+1} = O(x^{\log_2 \varepsilon})$ (for fixed ε), $F(x) = o(x^\alpha)$ follows from here by observing that $\varepsilon > 0$ was arbitrary (but n_ε depends on ε). Thus (9) follows. The proof of the Lemma is complete.

PROOF OF THE THEOREM. Observe that $H(\mathbf{o}) = 0$ by (2) and (3). Thus, according to Taylor's formula,

$$H\left(y_n, \ldots, y_{n+k}, \frac{1}{n}\right) = \sum_{l=1}^{m-1} \frac{1}{l!} \left(\sum_{j=0}^{k} y_{n+j}\partial_j + \frac{1}{n}\partial_{k+1}\right)^l H(\mathbf{o})$$
$$+ \frac{1}{m!} \left(\sum_{j=0}^{k} y_{n+j}\partial_j + \frac{1}{n}\partial_{k+1}\right)^m H\left(\theta y_n, \ldots, \theta y_{n+k}, \frac{\theta}{n}\right)$$

holds for some θ with $0 < \theta < 1$, provided n is large enough (so that the point $(y_n, \ldots, y_{n+k}, 1/n)$ belongs to a convex neighborhood of $\mathbf{o}$ in which H is m times continuously differentiable). The left-hand side here is zero according to (3). In view of the continuity of the mth derivatives of H at $\mathbf{o}$, (2) implies that the right-hand side will change only slightly if we replace the argument of H with $\mathbf{o}$ in the last term; estimating the magnitude of this change we obtain the following (note that the modified last term of the preceding formula being incorporated into the sum below, l now goes to m rather than $m - 1$):

$$\sum_{l=1}^{m} \frac{1}{l!}\left(\sum_{j=0}^{k} y_{n+j}\partial_j + \frac{1}{n}\partial_{k+1}\right)^l H(\mathbf{o}) = o\left(\sum_{j=0}^{k} |y_{n+j}|^m + n^{-m}\right) \tag{13}$$

as $n \to \infty$ (the function expressed as o may depend on k, m, and the bounds of the mth derivatives of H close to $\mathbf{o}$).

Using induction on m, we may assume here that

$$y_n = \sum_{l=1}^{m-1} c_l n^{-l} + \delta_n, \tag{14}$$

where

$$\delta_n = o(n^{-m+1}). \tag{15}$$

Indeed, for $m = 1$, (15) is justified by (2) and, for $m > 1$, it is justified by the induction hypothesis saying that (4) is valid with $m - 1$ replacing m. Substituting (14) into (13) and noting that $\delta_{n+j} = o(1)$ and $n^{-1}\delta_{n+j} = o(n^{-m})$ according to (15), we obtain that

$$\sum_{l=1}^{m-1} C_l n^{-l} + C_m' n^{-m} + \sum_{j=0}^{k} \delta_{n+j}\partial_j H(\mathbf{o}) = o\left(\sum_{j=0}^{k} |\delta_{n+j}|\right) + o(n^{-m}) \tag{16}$$

holds with some constants C_l, $1 \leqslant l < m$, and C_m'. (Some reflection shows that the first error term on the right is needed only in case $m = 1$, but we will not use this fact.) Notice here that C_m' depends solely on c_l for $1 \leqslant l < m$ and the ith order partial derivatives of H at $\mathbf{o}$ for $1 \leqslant i \leqslant m$. Substituting (15) into (16), it follows that $C_l = 0$ for $1 \leqslant l < m$; thus, (16) becomes

$$C_m' n^{-m} + \sum_{j=0}^{k} \delta_{n+j}\partial_j H(\mathbf{o}) = o\left(\sum_{j=0}^{k} |\delta_{n+j}|\right) + o(n^{-m}). \tag{17}$$

Choose

$$c_m = \frac{-C_m'}{\sum_{j=0}^{k} \partial_j H(\mathbf{o})} \tag{18}$$

(the denominator here is not zero according to (1)) and

$$f(n) = \delta_n - c_m n^{-m}. \tag{19}$$

In order to complete the proof of the Theorem, we have to show only that

$$f(n) = o(n^{-m}). \tag{20}$$

Indeed, if we show this, then (4) becomes valid in view of (14), (19) and (20).

To show (20), observe that according to (17)–(19) we have

$$\sum_{j=0}^{k} \left(f(n+j) + c_m\left((n+j)^{-m} - n^{-m}\right)\right)\partial_j H(\mathbf{o}) = o\left(\sum_{j=0}^{k} |\delta_{n+j}|\right) + o(n^{-m}).$$

The coefficient of c_m on the left is $o(n^{-m})$, and so we obtain

$$\sum_{j=0}^{k} \lambda_j f(n+j) = o\left(\sum_{j=0}^{k} |\delta_{n+j}|\right) + o(n^{-m}), \tag{21}$$

where $\lambda_j = \partial_j H(\mathbf{o})$. Note that, with this choice of λ_j, (7) has no roots of absolute value 1 in view of (1). By (19), we have

$$o\left(\sum_{j=0}^{k} |\delta_{n+j}|\right) = o\left(\sum_{j=0}^{k} |f(n+j)|\right) + o(n^{-m}) = \sum_{j=0}^{k} \eta_{nj} f(n+j) + o(n^{-m}),$$

where $\eta_{nj} \to 0$ as $n \to \infty$. Thus (21) becomes

$$\sum_{j=0}^{k} (\lambda_j - \eta_{nj}) f(n+j) = o(n^{-m}).$$

Using the above Lemma with $\lambda_{nj} = \lambda_j - \eta_{nj}$, we can conclude that (20) is indeed valid. This completes the proof of the Theorem.

3. An application to orthogonal polynomials. Let γ_n be the leading coefficient of the nth orthonormal polynomial on the real line with respect to the weight function $\exp(-x^6/6)$ ($n \geqslant 0$) and write $a_n = \gamma_{n-1}/\gamma_n$ ($a_n = 0$ for $n \leqslant 0$). Then the recurrence equation

$$\begin{aligned} a_n^2\big(a_{n-2}^2 a_{n-1}^2 + a_{n-1}^4 + 2a_{n-1}^2 a_n^2 + a_n^4 + a_{n-1}^2 a_{n+1}^2 \\ + 2a_n^2 a_{n+1}^2 + a_{n+1}^4 + a_{n+1}^2 a_{n+2}^2\big) = n \end{aligned} \tag{22}$$

is satisfied for all $n \geqslant 0$ (cf. the first equation on p. 6 in G. Freud [**1**]; Freud considers the weight function $|x|^\rho \exp(x^{-6})$, so his equation is a little different: he has a factor 6 on the left and a term $O(1)$ on the right). Freud shows that

$$\lim_{n\to\infty} a_n (10/n)^{1/6} = 1 \tag{23}$$

(see [**1**, p. 6]). Writing $F(a_{n+j}\colon -2 \leqslant j \leqslant 2)$ for the left-hand side of (22) and putting $y_n = a_n(10/n)^{1/6}$ and

$$H(x_j\colon -2 \leqslant j \leqslant 3) = F\left(x_j(1 + jx_3)^{1/6}\colon -2 \leqslant j \leqslant 2\right) - 1, \tag{24}$$

(22) becomes

$$H(y_{n-2}, y_{n-1}, y_n, y_{n+1}, y_{n+2}, 1/n) = 0, \tag{25}$$

which is analogous to (3), but the point $\mathbf{o}$ has to be replaced by $\mathbf{p} = \langle p_j\colon -2 \leqslant j \leqslant 3\rangle$ with $p_j = 1$ for $-2 \leqslant j \leqslant 2$ and $p_3 = 0$, in view of (23). As we have

$$\sum_{j=-2}^{2} \partial_j H(\mathbf{p})\, dx_j = 2dx_{-2} + 12dx_{-1} + 32dx_0 + 12dx_1 + 2dx_2, \tag{26}$$

the analogue of (1) is satisfied, because the coefficient of dx_0 is greater than the sum of (the absolute values of) all the other coefficients. Thus, by the above Theorem, y_n has an asymptotic expansion as described in (4), i.e., we have

$$a_n\left(\frac{10}{n}\right)^{1/6} = \sum_{l=0}^{m} c_l n^{-l} + o(n^{-m}) \tag{27}$$

as $n \to \infty$ for all integers m with some constants $c_0, c_1, \ldots, ; c_0 = 1$ by (23). (The Theorem gives the expansion of y_{n-2} in terms of n^{-l}, since y_{n-2} in (25) corresponds to y_n in (3); that is, the expansion of y_n is obtained in terms of $(n+2)^{-l} = n^{-l}(1+2/n)^{-l}$. Expansion (27) can then be derived by using the binomial expansion for $(1+2/n)^{-l}$.) We do not know whether or not the asymptotic expansion $\sum_{l=0}^{\infty} c_l n^{-l}$ with the coefficients c_l in (27) is convergent for any value of n (cf. [**2**, Chapter 11, p. 351 ff.] for a discussion of asymptotic expansions). A result analogous to (27) was obtained by J. S. Lew and D. A. Quarles, Jr., [**3**] for the weight function $\exp(-x^4/4)$, and the polynomials orthonormal with respect to this weight function are studied by Nevai [**5** and **6**].

We are going to show that

$$c_l = 0 \quad \text{for odd } l \tag{28}$$

in (27). This result is due to Rong Sheen (unpublished), who, assuming the validity of (27), established it in a way different from what follows. Using induction, assume that (28) is valid for every odd $l < m = 2r + 1$ $(r \geqslant 0)$. Write

$$y_n = t_n + c_m n^{-m} + o(n^{-m}),$$

where

$$t_n = \sum_{l=0}^{r} c_{2l} n^{-2l}, \tag{29}$$

which is justified according to (27) and the induction hypothesis, and substitute this into (25). Taking the first two terms (i.e., the constant term and the first order remainder) of the Taylor expansion of the left-hand side of the obtained equation at $\langle t_{n-2}, \ldots, t_{n+2}, 1/n\rangle$, we can conclude that

$$H(t_{n-2}, \ldots, t_{n+2}, 1/n) = -60 c_m n^{-m} + o(n^{-m}) \tag{30}$$

in view of (26) and the continuity of the derivatives of H. Writing $z = 1/n$, the left-hand side becomes

$$G(z) = F\left(\sum_{l=0}^{r} c_{2l} z^{2l}(1+jz)^{-2l+1/6} : -2 \leqslant j \leqslant 2\right) - 1$$

according to (29), with $n + j$ replacing n, and (24). As

$$F(x_j : -2 \leqslant j \leqslant 2) = F(x_{-j} : -2 \leqslant j \leqslant 2),$$

it is clear that $G(z) = G(-z)$; that is, the coefficients of odd powers of z in the Taylor expansion of G at $z = 0$ vanish. According to (30), the coefficient of z^m in this expansion is $-60c_m$; that is, $c_m = 0$. This completes the proof of (28).

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