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ON CONTINUITY OF SYMMETRIC RESTRICTIONS OF BOREL FUNCTIONS

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ABSTRACT. We prove that if X is a complete metric space dense-in-itself, Y is a compact metric space and F: $X \times X \setminus \{(x, x): x \in X\} \to Y$ is a Borel-measurable function such that $F(x_1, x_2) = F(x_2, x_1)$ for every $x_1, x_2 \in X$, $x_1 \neq x_2$, then there is a perfect subset P of X such that F is uniformly continuous on $P \times P \setminus \{(x, x): x \in P\}$. An immediate corollary of the above result is the following theorem proved by Bruckner, Ceder and Weiss: If F is a real continuous function defined on a perfect set $P \subset R$, there is a perfect subset Q of P such that $f|Q$ has a derivative (finite or infinite) at every point of Q.

For any set A we shall use the notation $A^2 = A \times A$ and $D(A) = \{(x, x): x \in A\}$. $\delta(A)$ will denote the diameter of A for $A \subset R$. A subset B of a cartesian square is called symmetric if $(y, x) \in B$ whenever $(x, y) \in B$.

We prove the following

THEOREM. *If X is a complete metric space dense-in-itself, Y is a compact metric space, and F: $X^2 \setminus D(X) \to Y$ is a Borel-measurable function such that $F(x_1, x_2) = F(x_2, x_1)$ for every $x_1, x_2 \in X$, $x_1 \neq x_2$, then there is a perfect (i.e., nonempty, closed, and dense-in-itself) subset P of X such that F is uniformly continuous on $P^2 \setminus D(P)$, whence it has a continuous extension to the whole P^2.*

The first part of the conclusion of this theorem, with "continuity" instead of "uniform continuity", is 2.3 in [**2**]. But, in fact, we want the continuous extension of F to $D(P)$ (what uniform continuity gives us), whence we can derive the following result of Bruckner, Ceder, and Weiss [**1**]:

Let f be a real continuous function on a perfect set P of real numbers. Then there exists a perfect set Q contained in P such that the restriction of f to Q has a derivative (finite or infinite) at every point of Q.

Indeed it is enough to put $X = P$, $Y = [0, 1]$ and

$$F(x_1, x_2) = \varphi\big((f(x_1) - f(x_2))/(x_1 - x_2)\big) \quad \text{for } x_1 \neq x_2,$$

where φ is any homeomorphism of R onto $(0, 1)$.

From now on $\mathscr{C}$ will denote only the Cantor discontinuum, i.e., the set of all numbers of $[0, 1]$ with triadic expansions in which digit one does not occur.

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We can assume without any loss of generality that $X = \mathscr{C}$, because every complete dense-in-itself metric space contains a copy of $\mathscr{C}$.

We can also assume $Y = \mathscr{C}$. Indeed, for every compact metric space Y there is a continuous surjection $f\colon \mathscr{C} \to Y$. Then $g(y) = \min\{c \in \mathscr{C} : f(c) = y\}$ is a Borel measurable function such that $f \circ g$ is the identity map on Y. Thus, if F satisfies the assumptions of our theorem, we can consider $g \circ F\colon X^2 \setminus D(X) \to \mathscr{C}$, for which, if the theorem is true for $\mathscr{C}$ as an image space, there is a perfect set $P \subset X$ such that $g \circ F$ is uniformly continuous on $P^2 \setminus D(P)$. Now it is enough to notice that $F = f \circ g \circ F$.

Our main tool is the following result of Galvin (announced in [**3** and **4**] in more general form):

MAIN LEMMA. *Let A and B be symmetric Borel subsets of $\mathscr{C}^2$ and $\mathscr{C}^2 \setminus D(\mathscr{C}) \subset A \cup B$. Then there exists a perfect subset P of $\mathscr{C}$ such that $P^2 \setminus D(P) \subset A$ or $P^2 \setminus D(P) \subset B$.*

The proof of the lemma can be found in [**2**]. It was based on Mycielski's theorem of [**5**] (2.2 in [**2**]).

PROOF OF THEOREM. We can assume with no loss of generality that F is continuous on $\mathscr{C}^2 \setminus D(\mathscr{C})$ (see 2.3 in [**2**]).

By the Main Lemma there exists a perfect set $Q \subset \mathscr{C}$ such that $F(Q^2 \setminus D(Q)) \subset [0, \frac{1}{2}]$ or $F(Q^2 \setminus D(Q)) \subset [\frac{1}{2}, 1]$. Assume $F(Q^2 \setminus D(Q)) \subset [0, \frac{1}{2}]$. Let $P_{(0)}$, $P_{(1)}$ be disjoint perfect sets contained in Q and $\delta(P_{(i)}) < 1$, $i = 0, 1$.

Again by the Main Lemma we can find disjoint perfect sets $P_{(i,0)}, P_{(i,1)} \subset P_{(i)}$, $i = 0, 1$, such that $F(P^2_{(i,j)} \setminus D(P_{(i,j)})) \subset [0, \frac{1}{4}]$ or $F(P^2_{(i,j)} \setminus D(P_{(i,j)})) \subset [\frac{1}{4}, \frac{1}{2}]$ and $\delta(P_{(i,j)}) < \frac{1}{2}$.

Continuing the choice of perfect sets $P_{(i_1,\ldots,i_n)}$ as above, we obtain $P_{(i_1,\ldots,i_{n+1})} \subset P_{(i_1,\ldots,i_n)}$, $P_{(i_1,\ldots,i_n,0)} \cap P_{(i_1,\ldots,i_n,1)} = \varnothing$, $\delta(P_{(i_1,\ldots,i_n)}) < 1/n$. Moreover,

$$F\Big(P^2_{(i_1,\ldots,i_n)} \setminus D\big(P_{(i_1,\ldots,i_n)}\big)\Big) \subset \left[\sum_{j=1}^{n} \alpha_j 2^{-j}, 2^{-n} + \sum_{j=1}^{n} \alpha_j 2^{-j}\right] \quad \text{for some } \alpha_j = 0 \text{ or } 1.$$

We shall prove that the set $P = \bigcap_{n=1} \bigcup_{(i_1,\ldots,i_n)} P_{(i_1,\ldots,i_n)}$ has the desired property, i.e. $F|P^2 \setminus D(P)$ is uniformly continuous.

Let us fix n. Let

$$A = \bigcup \left\{ P_{(i_1,\ldots,i_n)} \times P_{(j_1,\ldots,j_n)} : (i_1,\ldots,i_n) \neq (j_1,\ldots,j_n) \right\}.$$

A is compact, so $F|A$ is uniformly continuous, whence there is $\delta_0 > 0$ such that $|F(u) - F(v)| < 2^{-n}$ whenever $u, v \in A$ and the distance from u to v is less than δ_0.

Let δ_1 be the smallest distance between different sets $P_{(i_1,\ldots,i_n)} \times P_{(j_1,\ldots,j_n)}$.

The oscillation of F on each set $P^2_{(i_1,\ldots,i_n)} \setminus D(P_{(i_1,\ldots,i_n)})$ does not exceed 2^{-n}. Thus, $|F(u) - F(v)| \leqslant 2^{-n}$ whenever $u, v \in P^2 \setminus D(P)$ and the distance from u to v is less than $\min\{\delta_0, \delta_1\}$. This finishes the proof.

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