ENTROPY INCREASE AS A CONSEQUENCE
OF MEASURE INVARIANCE

C. C. BROWN

Abstract. An inequality, used in statistical mechanics for proving that entropy does
not decrease, is shown to hold for general σ-finite measure spaces. We comment
briefly on the corresponding Hilbert space result.

Let \((\Omega, \mathcal{A}, \mu)\) be a σ-finite measure space, \(\mathcal{F}_\mu\) the family of \(\mu\)-absolutely con-
tinuous measures on \(\mathcal{A}\), and \(T: \mathcal{F}_\mu \to \mathcal{F}_\mu\) a positive linear and monotone continuous
mapping; i.e. \(\sup, T(\nu_j) = T(\sup, \nu_j)\) for every monotone sequence, \(\nu_1 \leq \nu_2 \leq \cdots\), of
elements of \(\mathcal{F}_\mu\). If \(T\) preserves total measure, then every probability measure in \(\mathcal{F}_\mu\)
maps into a probability measure under \(T\). If \(\rho\) is a probability density with respect to
\((\Omega, \mathcal{A}, \mu)\), then the image of the corresponding probability distribution under \(T\) will
have a density \(\rho'\). A question of some interest in statistical mechanics concerns the
behavior of integrals of the form \(\int \phi(\rho(\omega)) \mu(d\omega)\). Suppose \(\phi\) is a convex function
defined in the nonnegative reals. Under what conditions is the inequality

\[
\int \phi(\rho(\omega)) \mu(d\omega) \geq \int \phi(\rho'(\omega)) \mu(d\omega)
\]

valid? The main case of interest for statistical mechanics is where \(\phi(x) = x \log x\)
\((x \geq 0)\), in which the integral corresponds to the Gibbs entropy [13].

A sufficient condition for the validity of the inequality is essentially the invariance
of \(\mu\) under \(T\). For finite \(\mu\) on a countable space \(\Omega\), this result can be found in the
book of Penrose [12]. The result appears to be attributable to M. J. Klein [10]. A
similar result has also been proved for finite \(\mu\) on a compact \(\Omega\) by J. Voigt [15,
Lemma 1.4].

Using an argument that is well known in the theory of probability in connection
with the representation of conditional expectations, it is possible to prove the
inequality for general σ-finite spaces. Slightly more generally, let \((\Omega, \mathcal{A}, \mu)\) and
\((\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mu})\) be σ-finite measure spaces, and \(T\) a monotone continuous positive linear
mapping from \(\mathcal{F}_\mu\) into the set of measures on \(\overline{\mathcal{A}}\). If \(T\mu = \overline{\mu}\), then \(Tv\) is \(\overline{\mu}\)-absolutely
continuous for every \(\nu \in \mathcal{F}_\mu\). If \(Tv(\overline{\Omega}) = \nu(\overline{\Omega})\) for every \(\nu \in \mathcal{F}_\mu\), then let \(\rho\) be a
probability density for \(\nu\) and \(\rho'\) a probability density for \(Tv\).
**Theorem.** For the mapping $T$ as above, let $\rho$ and $\rho'$ be the probability densities just defined, and let $\phi$ be a finite-valued convex function defined on $[0, \infty)$. If the integral, $H(\rho) := \int \phi(\rho(\omega)) \mu(d\omega)$, exists and is less than infinity, then the integral, $\overline{H}(\rho') := \int \phi(\rho'(\bar{\omega})) \bar{\mu}(d\bar{\omega})$, exists and is less than infinity. If both integrals exist in the extended sense, then $H(\rho) \geq \overline{H}(\rho')$.

**Proof.** Let $\mathcal{L}$ be the set of extended valued nonnegative measurable functions defined with respect to $\mathcal{A}$, and define $\mathcal{D}$ correspondingly for $(\Omega, \mathcal{A})$. For $g \in \mathcal{L}$ let $\nu_g \in \bar{\mathcal{L}}_\mu$ be the measure on $\mathcal{A}$ having the density $g$ with respect to $\mu$. Denoting by $\mathcal{L}_\mu$ the equivalence classes mod $\mu$ of elements of $\mathcal{L}$, the correspondence $g \mapsto dT(v)/d\mu$ defines a positive linear and monotone continuous mapping $T'$ from $\mathcal{L}_\mu$ into $\bar{\mathcal{L}}_\mu$, the set of $\bar{\mu}$-equivalence classes in $\mathcal{D}$. $T'$ is also a mapping of the same type from $\mathcal{L}_\mu$ into $\bar{\mathcal{L}}_\mu$. For $E \in \mathcal{A}$ let $I_E$ be the indicator function of $E$. $T'(I_E)$ is represented by a measurable function of $\bar{\omega} \in \bar{\Omega}$, and $I_\Omega \in T'(I_\Omega)$. Using this fact and an argument that is nearly the same as that in Breiman [2, Chapter 4] or Doob [3, Chapter I] for proving the existence of conditional probability kernels, if $f$ is a measurable mapping from $\Omega$ into a Borel space $(B, \mathcal{B})$, then we can construct a stochastic kernel $K_f$ from $(\bar{\Omega}, \bar{\mathcal{A}})$ to $(B, \mathcal{B})$ with the property

$$ K_f(F, \cdot) \in T'(I_{f \circ F}) \quad (F \in \mathcal{B}). $$

As in the conditional expectation case, one has

$$ \int \phi(y) K_f(dy, \cdot) \in T'(\psi \circ f) $$

for every nonnegative extended valued measurable function $\phi$ defined on $(B, \mathcal{B})$. Consequently,

$$ \rho' := \int |y| K_\rho(dy, \cdot) $$

is a density for $T_{\rho'}$. Since $\rho'$ is $\bar{\mu}$-almost everywhere finite, we can suppose that $K_\rho$ has been so chosen that $\rho'$ is everywhere finite. Thus $\int |y| K_\rho(dy, \cdot)$ is finite valued and the conditions for an application of Jensen’s inequality hold. For the convex function $\phi_+(y) := \max[\phi(y), 0]$ ($y \in [0, \infty)$), one has

$$ \phi_+(\rho') = \phi_+ \left( \int y K_\rho(dy, \cdot) \right) \leq \int \phi_+(y) K_\rho(dy, \cdot) $$

and

$$ \int \phi_+(\rho') \bar{\mu}(d\bar{\omega}) \leq \int \left( \int \phi_+(y) K_\rho(dy, \bar{\omega}) \right) \bar{\mu}(d\bar{\omega}) $$

$$ = \int T'(\phi_+ \circ \rho) \bar{\mu}(d\bar{\omega}) = \int \phi_+(\rho) \mu(d\omega). $$

Repeating this argument with $\phi$ in place of $\phi_+$, a proof of the theorem as stated is only a matter of technical details.

For a classical mechanical system, let $(\Omega, \mathcal{A}, \mu)$ be the phase space with the $\mu$-preserving family $\{T_t\}_{t \in \mathbb{R}}$ of one-to-one surjective transformations $T_t: \Omega \to \Omega$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
representing the system flow in $\Omega$. It is often possible to assume that $\mu$ is $\sigma$-finite on the $\sigma$-algebra $\mathcal{T}$ of invariant measurable sets in $\Omega$ (see, for example, [9]). It follows [9] that every initial probability density $\rho_0$ in $(\Omega, \mathcal{A}, \mu)$ has a time average which, in any reasonable definition of approach to equilibrium with time, is equal almost everywhere to the equilibrium density $\rho_\infty$ corresponding to $\rho_0$. The density $\rho_\infty$ is a conditional expectation $E(\rho_0|\mathcal{T})$ of $\rho_0$ under $\mu$ with respect to $\mathcal{T}$. Because $\mu$ is $\sigma$-finite on $\mathcal{T}$, $E(f|\mathcal{T})$ is also defined for every nonnegative measurable extended valued function $f$ on $\Omega$. The time average is therefore a restriction of the mapping $T'$ from $\mathcal{L} \to \mathcal{L}_\mu$ which derives from the mapping $T: \mathcal{F}_\mu \to \mathcal{F}_\mu$ given by $\nu_g \to \nu_{E(g|\mathcal{T})}$ ($g \in \mathcal{L}$). This mapping $T$ has the properties demanded by the theorem and is the only such mapping specializing to the average value map.

For quantum mechanical systems, a corresponding result has been proved by G. Lindblad [11]. The following easy consequence of Jensen's inequality seems to simplify Lindblad's proof considerably.

**Theorem.** Let $U \subset \mathbb{R}$ be an interval, $\phi$ a bounded convex function on $U$, $A_1, A_2, \ldots$ nonegative selfadjoint operators in a separable Hilbert space $\mathcal{H}$, with $A_1 + A_2 + \cdots = I =$ Identity operator, and $x_1, x_2, \ldots$ elements of $U$. If $\sum_a |x_a| \text{tr} A_a < \infty$ then $\phi(\sum_a x_a A_a)$ is defined as a bounded self adjoint operator in $\mathcal{H}$. If, furthermore, $\text{tr}[\sum_a \phi(x_a) A_a]$ exists less than infinity, then

$$\text{tr} \left[ \phi \left( \sum_a x_a A_a \right) \right] \leq \text{tr} \left[ \sum_a \phi(x_a) A_a \right].$$

**References**


Mathematics Institut, Freie Universität Berlin, Arnimallee 2-6, Berlin 33, West Germany