PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 3, March 1985

MAXIMAL SETS OF ORTHOGONAL MEASURES ARE NOT ANALYTIC

DAVID PREISS AND JAN RATAJ

ABSTRACT. By proving the theorem given in the title we answer a question posed by D. Mauldin at the conference on Measure Theory held in Oberwolfach in 1981.

In this note we study maximal sets of pairwise orthogonal probability measures on $\langle 0,1\rangle$. The existence of such sets is guaranteed by the Kuratowski-Zorn principle. Our theorem says that such sets cannot be Borel in the weak topology, or even analytic. This answers a problem posed by D. Mauldin at the Conference on Measure Theory held in Oberwolfach in 1981 (see [**2**]). To study maximal sets of pairwise orthogonal measures is useful in the light of general properties of orthogonal kernels; see [**2**] for a survey of these results and further references.

First let us introduce some notation. Assume that K is a compact metric space. $\mathscr{M}(K)$ denotes the locally convex space of all finite Borel signed measures on K with weak topology. $\mathscr{M}^+(K)$ denotes its subspace of all positive measures and $\mathscr{P}(K)$ its subspace of all probability measures. It is well known that $\mathscr{P}(K)$ is a compact metrizable space.

If $x \in K$ we denote by ε_x the Dirac probability measure concentrated at x. Let $\mathscr{D}(K)$ denote the set of all finite convex combinations of Dirac probabilities on K. $\mathscr{D}(K)$ is a dense subset of $\mathscr{P}(K)$.

We will also use the norm

$$\|\mu\| = \sup\{|\mu(A)|: A \text{ is a Borel subset of } K\} \quad \text{for } \mu \in \mathscr{M}(K).$$

The function $\mu \mapsto \|\mu\|$ is lower semicontinuous on $\mathscr{M}(K)$. Consequently, the function $(\mu, \nu) \mapsto \|\mu - \nu\|$ is lower semicontinuous on $\mathscr{P}(K) \times \mathscr{P}(K)$.

If L is a compact subspace of K and $B \subseteq \mathscr{P}(K)$, we define the set of restrictions $B \restriction L = \{\mu \restriction L: \mu \in B\}$.

Now consider the special case $K = \langle 0,1\rangle$. Assume that $r \in \mathbf{N}$, $I_1,\ldots,I_r$ are disjoint compact intervals in $\langle 0,1\rangle$; by an interval we always mean a nondegenerate interval, further, that $\lambda_1,\ldots,\lambda_r$ are positive numbers such that $\sum_{i=1}^r \lambda_i = 1$ and

Received by the editors June 17, 1983 and, in revised form, August 22, 1983.
1980 *Mathematics Subject Classification*. Primary 28A33; Secondary 60B05.

$\Delta > 0$. Then we put

$$(1)\quad U(r; I_1,\dots,I_r;\lambda_1,\dots,\lambda_r;\Delta) = \{\mu \in \mathscr{P}(\langle 0,1\rangle): \mu(\operatorname{int} I_i) > \lambda_i - \Delta, \mu(I_i) \leqslant \lambda_i + \Delta, i = 1,\dots,r\}.$$

If $\mu \in \mathscr{D}(\langle 0,1\rangle)$, $\mu = \sum_{i=1}^r \lambda_i \varepsilon_{x_i}$, then the system of all sets

$$U(r; I_1,\dots,I_r; \lambda_1,\dots,\lambda_r; \Delta)$$

such that $x_i \in \operatorname{int} I_i$, $i = 1,\dots,r$, and $\Delta > 0$ forms a basis of open neighbourhoods of μ. Consequently, every open set in $\mathscr{P}(\langle 0,1\rangle)$ contains an open set of the type (1).

For the proof we need the Banach-Mazur game as an instrument for proving residuality (see [3]). Assume that X is a complete metric space and $M \subseteq X$. The Banach-Mazur game is played by two players, (A) and (B). In the first step, (A) puts $U_1 = X$ and (B) chooses an open set $U_1' \subseteq U_1$. In the nth step (A) chooses an open set $U_n \subseteq U_{n-1}'$ and (B) chooses an open $U_n' \subseteq U_n$. This defines a nonincreasing sequence of open sets. If their intersection contains an element of M, player (A) wins, in the opposite case (B) wins. It is well known that (A) has a winning strategy in this game if and only if M is residual in X; the proof in [3] easily extends for the general case.

Our main result is given in the following

THEOREM. *No maximal set of pairwise orthogonal Borel probability measures on the interval $\langle 0,1\rangle$ is analytic.*

For the proof we will use two auxiliary lemmas:

LEMMA 1. *Assume that $K \subseteq \langle 0,1\rangle$ is a finite union of compact disjoint intervals, $V \subseteq \mathscr{P}(\langle 0,1\rangle)$ an open set, and $\varepsilon \in (0,1)$. Then the set $Y = \{\zeta \in \mathscr{P}(K): \varepsilon \cdot \zeta \in V \restriction K\}$ is open in $\mathscr{P}(K)$.*

PROOF. Let ζ_0 be an arbitrary measure in Y and let $\mu_0 \in V$ be an extension of $\varepsilon \cdot \zeta_0$. Clearly, if $\zeta \in \mathscr{P}(K)$ is sufficiently close to ζ_0 then the measure μ, defined by $\mu \restriction K = \varepsilon \cdot \zeta$, $\mu \restriction \langle 0,1\rangle \setminus K = \mu_0 \restriction \langle 0,1\rangle \setminus K$, extends $\varepsilon \cdot \zeta$ and is arbitrarily close to μ_0.

LEMMA 2. *Assume that K is the compact set from Lemma 1, $V \subseteq \mathscr{P}(\langle 0,1\rangle)$ an open set, $T \subseteq V$ is dense and open in V, $U \subseteq \mathscr{P}(K)$ is an open set, $\varepsilon \in (0,1)$, and $\Delta > 0$. Assume further that for every $\zeta \in U$ the measure $\varepsilon \cdot \zeta$ can be extended into V (or, equivalently $\varepsilon \cdot \zeta \in V \restriction K$). Then the set $Y = \{\zeta \in U$: there exists $t \in (\varepsilon - \Delta, \varepsilon + \Delta)$, such that $t \cdot \zeta \in T \restriction K\}$, is open and dense in U.*

PROOF. From Lemma 1 we see that Y is open. We prove that it is also dense in U. Let U_0 be any nonempty open subset of U. There is $\zeta_0 \in U_0$ such that (i) $\zeta_0(\operatorname{bdry} K) = 0$. By the assumption, $\varepsilon \cdot \zeta_0$ can be extended at some $\mu_0 \in V$. Let μ_n be any sequence of elements of T converging to μ_0. By (i), $\mu_n \restriction K \to \mu_0 \restriction K = \varepsilon \cdot \zeta_0$. If $\zeta_n = (\mu_n(K))^{-1} \cdot \mu_n \restriction K$, then $\zeta_n \to \zeta_0$. Since $\mu_n(K) \to \varepsilon$, $\zeta_n \in U_0 \cap Y$ for all sufficiently large n.

The next lemma is the main step of the proof of our theorem.

LEMMA 3. *Assume that* $K \subseteq \langle 0,1\rangle$ *is the compact set from Lemma* 1, $0 \subseteq \mathscr{P}(K)$ *is a nonempty open set in* $\mathscr{P}(K)$, $B \subseteq \mathscr{P}(\langle 0,1\rangle)$ *is residual in an open nonempty set* $G \subseteq \mathscr{P}(\langle 0,1\rangle)$ *and* $\varepsilon \in (0,1)$. *Assume further that*

(1) *there exists* $t_0 > \varepsilon$ *such that, for every* $\zeta \in 0$, $t_0 \cdot \zeta \in G \restriction K$.

Then the set $X = \{\zeta \in 0$: *there is* $t > \varepsilon$ *such that* $t \cdot \zeta \in B \restriction K\}$ *is residual in* 0.

PROOF. There are dense, open subsets H_n in G such that $\bigcap_{n=1}^{\infty} H_n \subseteq B$.

Let us fix a number $0 < \delta < |t_0 - \varepsilon|$. We prove the residuality of the set X in 0 by describing a winning strategy in the Banach-Mazur game. We use the same notation as in the beginning of this text. Put $U_1 = 0$. Player (B) chooses open sets $U_i' \subseteq 0$ such that

(i) $U_i' \subseteq U_i$, $i = 1,2,\ldots$, and (A) chooses open sets $U_i \subseteq U_{i-1}'$, $i = 2,3,\ldots$. We will even require that

(ii) $\overline{U}_i \subseteq U_{i-1}$, $i = 2,3,\ldots$. We will describe the moves of (A), no matter how (B) played in the preceding steps. In addition, we will construct open sets $V_i \subseteq \mathscr{P}(\langle 0,1\rangle)$, $i = 1,2,\ldots$, and numbers $\varepsilon_i > 0$, $i = 1,2,\ldots$, with the following properties:

(iii) $\overline{V}_i \subseteq V_{i-1} \cap H_i$, $i = 2,3,\ldots$,

(iv) $|\varepsilon_i - \varepsilon_{i-1}| < \delta \cdot 2^{-i}$, $i = 2,3,\ldots$,

(v) if $\zeta \in U_i$, then $\varepsilon_i \cdot \zeta \in V_i \restriction K$, $i = 1,2,\ldots$,

(vi) $\operatorname{diam} V_i < i^{-1}$, $i = 2,3,\ldots$,

(vii) if $\mu \in V_i$ then $\mu(\operatorname{bdry} K) < i^{-1}$, $i = 2,3,\ldots$.

CONSTRUCTION. We put $V_1 = G$ and $\varepsilon_1 = t_0$. Then, by the assumption (1) of this lemma, the objects U_1, V_1, ε_1 satisfy the condition (v). The remaining conditions have no relation to the case $i = 1$.

Assume that open sets $U_1, U_1', V_1, \ldots, U_{n-1}, U_{n-1}', V_{n-1}$ and positive numbers $\varepsilon_1, \ldots, \varepsilon_{n-1}$ have already been defined and that the requirements (i)–(vii) hold. By Lemma 2 the set $X_n = \{\zeta \in U_{n-1}'$: there is $s \in (\varepsilon_{n-1} - \delta \cdot 2^{-n}, \varepsilon_{n-1} + \delta \cdot 2^{-n})$ such that $s \cdot \zeta \in (V_{n-1} \cap H_n) \restriction K\}$, is dense and open in U_{n-1}'. Take $\zeta_n \in X_n$ such that $\zeta_n(\operatorname{bdry} K) = 0$ and take $\varepsilon_n \in (\varepsilon_{n-1} - \delta \cdot 2^{-n}, \varepsilon_{n-1} + \delta \cdot 2^{-n})$ such that $\varepsilon_n \cdot \zeta_n$ can be extended to some $\mu_n \in V_{n-1} \cap H_n$; requirement (iv) holds for ε_n. There exists an open set $V_n \subseteq \mathscr{P}(\langle 0,1\rangle)$ such that $\mu_n \in V_n$ and the requirements (iii), (vi), (vii) hold. By Lemma 1, if U_n is a sufficiently small neighbourhood of ζ_n, the requirements (ii), (v) also hold.

Finally we verify that our strategy is really winning. Let ζ be a measure from the intersection $\bigcap_{n=1}^{\infty} U_n$; we show that $\zeta \in X$. By (iv), the sequence ε_n converges to some $\bar{\varepsilon} > \varepsilon$. By (v), $\varepsilon_n \cdot \zeta$ can be extended to some $\mu_n \in V_n$, $n = 1,2,\ldots$, and, by (iii) and (vi), there exists a measure $\mu \in \bigcap V_n \subseteq \bigcap H_n \subseteq B$ such that $\mu_n \to \mu$. By (vii), $\mu(\operatorname{bdry} K) = 0$, hence $\mu_n \restriction K \to \mu \restriction K$. On the other hand, the sequence $\mu_n \restriction K = \varepsilon_n \cdot \zeta$ converges to $\bar{\varepsilon} \cdot \zeta$; consequently, $\mu \restriction K = \bar{\varepsilon} \cdot \zeta$. In other words, $\mu \in B$ is an extension of $\bar{\varepsilon} \cdot \zeta$.

We now see that this strategy of the Banach-Mazur game with the set $X \subseteq 0$ is winning for (A); hence, by the Theorem mentioned in the beginning of this paper, X is residual in 0.

LEMMA 4. *Assume that $G \subseteq \mathscr{P}(\langle 0,1\rangle)$ is a nonempty open convex set. Let $G_1 \subseteq G$ and $G_2 \subseteq G$ be open nonempty sets such that $G \subseteq \overline{G}_1 \cup \overline{G}_2$, and let $\tau \in (0,1)$. Then*

(a) *there exists measures $\mu_0 \in \overline{G}_1 \cap G$ and $\nu_0 \in G_2$ such that $\tau \cdot \mu_0 \leqslant \nu_0$, and*

(b) *there exists a compact $K \subseteq (\langle 0,1\rangle)$ which is a finite union of compact disjoint intervals and there exists a nonempty open subset $0 \subseteq \mathscr{P}(K)$ such that, for every $\zeta \in 0$, $\tau \cdot \zeta \in G_1 \restriction K \cap G_2 \restriction K$.*

PROOF. (a) Let $\mu \in G_1$ and $\nu \in G_2$ be arbitrary measures. Put (i)

$$\xi_k = \tau^k \cdot \mu + (1 - \tau^k) \cdot \nu, \quad k = 0,1,2,\ldots.$$

Clearly $\xi_k \in G$ for every k, because ξ_k are convex combinations of μ and ν. Let $k_0 \geqslant 1$ be the least natural number such that $\xi_{k_0} \in G_2$; the existence of such a number follows from the fact that $\xi_k \to \nu \in G_2$. We denote $\mu_0 = \xi_{k_0-1}$ (clearly $\mu_0 \in \overline{G}_1 \cap G$) and $\nu_0 = \xi_{k_0}$. From (i) we get

$$\nu_0 = \tau^{k_0} \cdot \mu + (1 - \tau^{k_0}) \cdot \nu = \tau \cdot \mu_0 + (1 - \tau) \cdot \nu,$$

hence $\tau \cdot \mu_0 \leqslant \nu_0$.

(b) We use the measures μ_0, ν_0 from (a). There exists a sequence μ_n, $n = 1,2,\ldots$, in $G_1 \cap \mathscr{D}(\langle 0,1\rangle)$ such that $\mu_n \to \mu_0$. Define $\nu_n = \tau \cdot \mu_n + (\nu_0 - \tau \cdot \mu_0)$, $n = 1,2,\ldots$. We see that $\nu_n \to \nu_0$, hence there exists n_0 such that $\nu_{n_0} \in G_2$. Let μ_{n_0} have the form $\mu_{n_0} = \sum_{i=1}^r \lambda_i \cdot \varepsilon_{x_i}$, where $r \in \mathbf{N}$, $x_i \in \langle 0,1\rangle$, $\lambda_i > 0$, $i = 1,\ldots,r$, $x_i \neq x_j$ for $i \neq j$ and $\sum_{i=1}^r \lambda_i = 1$. Let $\tilde{\mu} = \mu_{n_0}$. The measure $\nu_0 - \tau \cdot \mu_0$ can be approximated by a combination of Dirac measures $\sum_{i=r+1}^s \xi_i \cdot \varepsilon_{x_i}$ (where $x_i \in \langle 0,1\rangle \setminus \{x_1,\ldots,x_r\}$, $\xi_i > 0$, $i = r+1,\ldots,s$, $x_i \neq x_j$ for $i \neq j$) such that $\tilde{\nu} = \tau \cdot \mu_0 + \sum_{i=r+1}^s \xi_i \cdot \varepsilon_{x_i} \in G_2$.

There exist compact disjoint intervals $I_i \subseteq \langle 0,1\rangle$, $x_i \in \operatorname{int} I_i$, $i = 1,\ldots,s$, and a number $\Delta > 0$ such that neighbourhoods

$$U(\tilde{\mu}) = U(r; I_1,\ldots,I_r; \lambda_1,\ldots,\lambda_r; \Delta)$$

and

$$U(\tilde{\nu}) = U(s; I_1,\ldots,I_s; \tau \cdot \lambda_1,\ldots,\tau \cdot \lambda_r, \xi_{r+1},\ldots,\xi_s; \tau \cdot \Delta)$$

fulfill the inclusions $U(\tilde{\mu}) \subseteq G_1$ and $U(\tilde{\nu}) \subseteq G_2$. Take compact intervals $K_i \subseteq \operatorname{int} I_i$, $i = 1,\ldots,r$. Put $K = K_1 \cup \cdots \cup K_r$ and

$$0 = \{\zeta \in \mathscr{P}(K) : \zeta(K_i) \in (\lambda_i - \Delta/2, \lambda_i + \Delta/2), i = 1,\ldots,r\}.$$

We now see that for every $\zeta \in 0$ the measure $\tau \cdot \zeta$ can be extended into both neighbourhoods $U(\tilde{\mu})$, $U(\tilde{\nu})$, and hence into both sets G_1 and G_2.

PROOF OF THE THEOREM. Let A be a maximal set of pairwise orthogonal Borel probability measures in $\mathscr{P}(\langle 0,1\rangle)$. Assume, to the contrary, that A is analytic.

For every $k \in \mathbf{N}$ let E_k denote the space of all k-element subsets of A. We equip E_k with its usual topology (generated, e.g., by the Hausdorff metric). Clearly, each E_k is a continuous image of an open subset of the Cartesian product A^k, hence it is analytic.

Whenever $k \in \mathbf{N}$ and $\varepsilon \in (0,1)$, the set of all $(\mu, j) \in \mathscr{P}(\langle 0,1\rangle) \times E_k$ such that $\|\mu - \nu\| \leqslant \varepsilon$ for every $\nu \in j$ is closed in $\mathscr{P}(\langle 0,1\rangle) \times E_k$. It follows that for every analytic set $C \subseteq E_k$ the set $\{(\mu, j) \in \mathscr{P}(\langle 0,1\rangle) \times C;\ \|\mu - \nu\| < \varepsilon$ for every $\nu \in j\}$ is analytic and hence its first projection $H_{k,\varepsilon}(C)$ is also analytic. In addition, let $\tau \in (0, \varepsilon)$. Define $F^{\tau}_{k,\varepsilon}(C) = H_{k,\varepsilon-\tau}(C) - H_{k+1,\varepsilon}(E_{k+1})$ and $U^{\tau}_{k,\varepsilon} = F^{\tau}_{k,\varepsilon}(E_k)$. Then:

(1) Whenever $k \in \mathbf{N}$, $\varepsilon \in (0,1)$, $\varepsilon > \tau > 0$ and $C \subseteq E_k$ is analytic, the set $F^{\tau}_{k,\varepsilon}(C)$ is a difference of two analytic sets and hence it has the property of Baire.

(2) If $C_1 \subseteq E_k$ and $C_2 \subseteq E_k$ are disjoint sets, $\mu_1 \in F^{\tau}_{k,\varepsilon}(C_1)$ and $\mu_2 \in F^{\tau}_{k,\varepsilon}(C_2)$, then $\|\mu_1 - \mu_2\| \geqslant \tau$. (Otherwise we could find $\{\nu_1, \ldots, \nu_k\} \in C_1$ and $\{\bar{\nu}_1, \ldots, \bar{\nu}_k\} \in C_2$ such that $\|\mu_1 - \nu_i\| < \varepsilon$ and $\|\mu_1 - \bar{\nu}_i\| \leqslant \|\mu_1 - \mu_2\| + \|\mu_2 - \bar{\nu}_i\| < \varepsilon$ for every $i = 1, \ldots, k$. The disjointedness of C_1 and C_2 would imply that $\mu_1 \in H_{k+1,\varepsilon}(E_{k+1})$.)

(3) Since A is a maximal set of orthogonal measures, for every $\nu \in \mathscr{P}(\langle 0,1\rangle)$ the set $\{\mu \in A;\ \|\nu - \mu\| < 1\}$ is nonempty and for every $t \in (0,1)$ the set $\{\mu \in A;\ \|\nu - \mu\| < t\}$ is finite. We find $\varepsilon \in (0,1)$ such that $\|\nu - \mu\| < \varepsilon$ for some $\mu \in A$. Let $j = \{\mu \in A;\ \|\nu - \mu\| < \varepsilon\}$. Since j is finite, there is $\tau \in (0, \varepsilon)$ such that $\|\mu - \nu\| < \varepsilon - \tau$ for each $\mu \in j$. Thus $\nu \in U^{\tau}_{k,\varepsilon}$, where k is the number of elements of j. Consequently,

$$\mathscr{P}(\langle 0,1\rangle) = \bigcup_{\varepsilon \in (0,1)} \bigcup_{k=1}^{\infty} \bigcup_{\tau \in (0,\varepsilon)} U^{\tau}_{k,\varepsilon}.$$

We now prove that $U^{\tau}_{k,\varepsilon}$ is of first category for every $k \in \mathbf{N}$, $\varepsilon \in (0,1)$, $\varepsilon > \tau > 0$. Assume, to the contrary, that $U^{\tau}_{k,\varepsilon}$ is of second category. Then, by the property of Baire, it is residual in some nonempty open set (see [**1**]). We can assume that $U^{\tau}_{k,\varepsilon}$ is residual in a convex nonempty open set $G \subseteq \mathscr{P}(\langle 0,1\rangle)$. Denote by Z the maximal open set in E_k such that $F^{\tau}_{k,\varepsilon}(Z)$ is of first category in G. We see that $F = E_k \setminus Z$ has at least two different elements, because the set $F^{\tau}_{k,\varepsilon}(\{j\})$ is nowhere dense for every $j \in E_k$. Take a partition $F = F_1 \cup F_2$, where F_1 and F_2 are Borel subsets of F with nonempty interiors. Put $A_1 = F^{\tau}_{k,\varepsilon}(F_1)$, $A_2 = F^{\tau}_{k,\varepsilon}(F_2)$. Clearly, A_1 and A_2 are both of second category in G; by (1) they have the property of Baire, and hence they are residual in some open sets. Denote by G_1, G_2 the maximal open subsets in G, such that A_1, A_2 are residual in G_1, G_2, respectively. Clearly $G \subseteq \overline{G}_1 \cup \overline{G}_2$. By Lemma 4 there exists a compact subset $K \subseteq \langle 0,1\rangle$ which is a finite union of compact disjoint intervals, and a nonempty open set $0 \subseteq \mathscr{P}(K)$ such that for every $\zeta \in 0$ the measure $(1-\tau) \cdot \zeta$ can be extended into G_1 and G_2. By Lemma 3, the sets

$$X_1 = \{\zeta \in 0\colon \text{there is } t_1 > 1 - \tau \text{ such that } t_1 \cdot \zeta \in A_1 \restriction K\}$$

and

$$X_2 = \{\zeta \in 0\colon \text{there is } t_2 > 1 - \tau \text{ such that } t_2 \cdot \zeta \in A_2 \restriction K\}$$

are residual in 0. Hence their intersection is nonempty and there exist $\zeta \in 0$, $\mu_1 \in A_1$, $\mu_2 \in A_2$ and $t_1, t_2 > 1 - \tau$ such that $\mu_1 \restriction K = t_1 \cdot \zeta$ and $\mu_2 \restriction K = t_2 \cdot \zeta$. Hence $\|\mu_1 - \mu_2\| < \tau$. But by (2) we see that $\|\mu_1 - \mu_2\| \geqslant \tau$. This contradiction proves that $U^{\tau}_{k,\varepsilon}$ is of first category.

In (3) it suffices to take only countable unions, hence the whole space $\mathscr{P}(\langle 0,1\rangle)$ is of first category. This contradiction proves our Theorem.

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FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKA 83, 186 000 PRAGUE 8, CZECHOSLOVAKIA