MULTIPLIERS FOR EIGENFUNCTION EXPANSIONS
OF SOME SCHRÖDINGER OPERATORS

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ABSTRACT. Let $G$ be a graded nilpotent Lie group and let $L$ be a positive Rockland operator on $G$. Let $E_{\lambda}$ denote the spectral resolution of $L$ on $L^2(G)$. A sufficient condition is given under which a function $m$ on $\mathbb{R}^+$ is an $L^p$-multiplier for $L$, $1 < p < \infty$; that is, $\| \int_0^\infty m(\lambda) \, dE_{\lambda} f \|_p \leq C_p \| f \|_p$ for a constant $C_p$, $f \in L^p(G) \cap L^2(G)$. Then the same is done for an operator $\pi(L)$, where $\pi$ is a unitary representation of $G$ induced from a unitary character of a normal connected subgroup $H$ of $G$. Hence the case of the Hermite operator $-d^2/dx^2 + x^2$ is covered and an $L^p$-multiplier theorem for classical Hermite expansions is obtained.

1. Let $L$ be an essentially selfadjoint on its domain, positive, densely defined operator on $L^2(X)$, where $X$ is a measure space. Let $E_{\lambda}$ be a spectral resolution of the identity for which

$$Lf = \int_0^\infty \lambda \, dE_{\lambda} f, \quad f \in \text{Dom}(L).$$

If $m$ is a bounded measurable function on $\mathbb{R}^+$, we write $m(L)$ for the “multiplier operator”

$$m(L)f = \int_0^\infty m(\lambda) \, dE_{\lambda} f, \quad f \in L^2(X),$$

which is bounded on $L^2(X)$.

The question about conditions on $m$ which guarantee that $m(L)$ is bounded on $L^p(X)$ has been raised by E. M. Stein [9] and discussed in a number of papers afterwards. Thus we say that a function $m$ is an $L^p$-multiplier for the operator $L$ if there exists a constant $C_p$ such that for all $f$ in $L^p(X) \cap L^2(X)$

$$\|m(L)f\|_p \leq C_p \| f \|_p.$$

In [1] A. Bonami and J. L. Clerc present a technique, based on investigation of some $g$-functions of Paley-Littlewood type, which allows them to obtain an $L^p$-multiplier theorem in the case when $L$ is the Laplace-Beltrami operator on a compact, riemannian manifold. An estimate due to L. Hörmander (cf. [1, p. 259]) plays here the crucial role.
Recently, A. Hulanicki and J. W. Jenkins [6, 7, 8] have obtained similar estimates for a class of operators which are of different types and include, among others, operators \(-\Delta + V\), where \(\Delta\) is the Laplacian on \(\mathbb{R}^4\) and \(V\) is a sum of squares of polynomials.

In this paper we observe that a method used by Bonami and Clerc can be applied to results of Hulanicki and Jenkins.

We obtain a stronger version of the Hulanicki-Stein multiplier theorem for a sub-Laplacian (cf. [5]) and for a positive Rockland operator (cf. [6]), and we produce new multiplier theorems for the eigenfunction expansions of Schrödinger type operators.

We illustrate our results by the following theorem concerning the Hermite expansion.

**THEOREM 1.** There exists an integer \(N\) such that if \(m \in C^N(\mathbb{R}^+)\) satisfies the conditions

\[
|m(\lambda)| \leq M, \quad \sup_{A > 0} \frac{1}{A} \int_0^A \lambda^k |m^{(k)}(\lambda)| d\lambda \leq M, \quad k = 1, \ldots, N,
\]

then for every \(1 < p < \infty\) the sequence \(m(2j + 1), j = 0, 1, \ldots, \) is a multiplier on \(L^p(\mathbb{R})\) with respect to the Hermite expansion; that is, for all \(f \in L^p(\mathbb{R}) \cap L^2(\mathbb{R})\)

\[
\left\| \sum m(2j + 1) \langle h_j, f \rangle h_j \right\|_p \leq C_p \|f\|_p
\]

with a constant \(C_p\).

Here \(h_j\) is a \(j\)th Hermite function and

\[
\langle h_j, f \rangle = \int_{\mathbb{R}} h_j(x) f(x) \, dx.
\]

2. Let \(E_{\lambda}\) be a positive (i.e. \(E_{\lambda} = 0\) for \(\lambda \leq 0\)) spectral resolution of the identity on \(L^2(\mathbb{X})\). Denote by \(S_{\mathbb{R}}^\delta, R > 0\), the \(\delta\)-Riesz mean

\[
S_{\mathbb{R}}^\delta f = \int_0^R \left(1 - \frac{\lambda}{R}\right)^\delta dE_{\lambda} f, \quad f \in L^2(\mathbb{X}).
\]

We suppose that \(E_{\lambda}\) has the following properties:

(a) For \(p > 1\) and a positive integer \(\delta\) the Marcinkiewicz-Zygmund property holds, i.e.

there exists a constant \(B_p\) such that for an arbitrary sequence of positive numbers \(R_j\) and for an arbitrary sequence of functions \(f_j \in L^p(\mathbb{X}) \cap L^2(\mathbb{X})\) we have

\[
\left\| \left( \sum |S_{\mathbb{R}}^\delta f_j|^2 \right)^{1/2} \right\|_p \leq B_p \left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_p.
\] (2.1)

(b) The operators \(T' = \int_0^\infty \exp(-\lambda t) \, dE_{\lambda}\) are contractions on all \(L^p(\mathbb{X}), 1 \leq p \leq \infty\).
The following theorem is essentially due to Bonami and Clerc (cf. [1, p. 260]; the details of the proof extracted from [1] and adapted to our situation can be found in [10]).

**Theorem A.** Let $E_\lambda$ be a positive spectral resolution of the identity which satisfies (a) and (b). If a function $m \in C^{\delta+1}(\mathbb{R}^+)$ satisfies the conditions

$$|m(\lambda)| \leq M, \quad \sup_{\lambda > 0} \frac{1}{A} \int_0^A |m^{(j)}(\lambda)| d\lambda \leq M, \quad j = 1, \ldots, \delta + 1,$$

then $m$ is an $L^p$-multiplier for $E_\lambda$; that is, there exists a constant $C_p$ such that

$$\left\| \int_0^\infty m(\lambda) \, dE_\lambda f \right\|_p \leq C_p \|f\|_p, \quad f \in L^p(X) \cap L^2(X).$$

The constant $C_p$ does not depend on the function $m$ but only on $M$.

In the sequel we deal with spaces of homogeneous type, in the sense of [2] only.

Let us recall that a topological space $X$ equipped with a continuous pseudometric $\rho$ and with a measure $\mu$, which, for a constant $C$, satisfies

$$(2.2) \quad \mu(B_r(y)) \leq C \mu(B_{r/2}(x)), \quad x, y \in X, r > 0,$$

is a space of homogeneous type and the Hardy-Littlewood maximal function

$$m^*f(x) = \sup_{r > 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y)| d\mu(y)$$

is of weak type $(1,1)$ (cf. [2]). Here $B_r(x)$ denotes the ball $B_r(x) = \{ y : \rho(x, y) < r \}$.

Moreover, the following Marcinkiewicz-Zygmund type inequality due to C. Fefferman and E. M. Stein is valid (cf. [4]; see also [1]).

**Theorem B.** Let $\mu$ be a measure and $\rho$ a pseudometric on $X$ such that (2.2) holds. Then for every $1 < p < \infty$ there exists a constant $D_p$, such that for an arbitrary sequence of functions $f_i$ we have

$$\left\| \left( \sum |m^*f_i|^2 \right)^{1/2} \right\|_p \leq D_p \left\| \left( \sum |f_i|^2 \right)^{1/2} \right\|_p.$$

Observe that if the spectral measure $E_\lambda$ on $L^2(X)$ satisfies the estimate

$$(2.3) \quad \sup_{R > 0} \left| S_R^\delta f \right| \leq C m^*f,$$

then, in virtue of Theorem B, $E_\lambda$ has for every $p$, $1 < p < \infty$, the Marcinkiewicz-Zygmund property (2.1).

3. Let $G$ be a graded nilpotent Lie group of [5]. $G$ endowed with the Haar measure and the pseudometric generated by the homogeneous norm is of homogeneous type (cf. [5]).

Let $L$ be a positive Rockland operator on $G$ that is a homogeneous, left-invariant differential operator such that $\pi(L)$ is injective on $C^\infty$-vectors for every irreducible, nontrivial unitary representation $\pi$ of $G$. It is known (cf. [5]) that $L$ is hypoelliptic, and as such it is essentially selfadjoint on $L^2(G)$. 
Let $E_\nu(\lambda)$ be a spectral resolution of the identity for $L$. The operators $T^t = \int_0^\infty \exp(-t\lambda) \, dE_\nu(\lambda)$ are contractions on all $L^p(G)$, $1 \leq p \leq \infty$ (cf. [5]).

As we have mentioned before, Hulanicki and Jenkins have shown (cf. [6, 8]; see also [7]) that there exists an integer $N(L)$ such that

$$\sup_{R > 0} \left| \mathcal{S}^R_{N(L)} f \right| \leq Cm^*_R f,$$

where $m^*_R$ is the Hardy-Littlewood maximal function on $G$ and $\mathcal{S}^R_{N(L)}$, $R > 0$, are Riesz means corresponding to $E_\nu(\lambda)$.

Thus we can state the following

**Proposition 1.** Let $L$ be a positive Rockland operator and let $E_\nu(\lambda)$ be a spectral resolution of $L$. Let $N(L)$ be an integer such that (3.1) holds. If $m \in C^{N(L)+1}(\mathbb{R}^+)$ satisfies the conditions

$$|m(\lambda)| \leq M, \quad \sup_{A>0} \frac{1}{A} \int_A^A \lambda^k |m^{(k)}(\lambda)| \, d\lambda \leq M, \quad k = 1, \ldots, N(L) + 1,$$

for a constant $M$, then for every $p$, $1 < p < \infty$, the function $m$ is an $L^p$-multiplier for the operator $L$.

4. Let $G$ be an arbitrary nilpotent Lie group and let $\pi$ be a representation of $G$ induced from a unitary character of a normal connected subgroup $H$ of $G$. The operators $\pi(x)$, $x \in G$, act on $L^2(G/H)$.

Let $X_1, \ldots, X_k$ be elements which generate the Lie algebra of $G$. Put

$$L = \sum_{j=1}^k (-1)^{n_j} X_j^{2n_j},$$

where $n_j$, $j = 1, \ldots, k$, are arbitrary positive integers. Then $\pi(L)$ is a positive essentially selfadjoint operator on $L^2(G/H)$. Denote by $E_{\pi(L)}(\lambda)$ the spectral resolution of $\pi(L)$.

**Proposition 2.** Let $L$ and $E_{\pi(L)}$ be as above. Then there exists an integer $N$ such that if $m \in C^{N}(\mathbb{R}^+)$ satisfies the conditions

$$|m(\lambda)| \leq M, \quad \sup_{A>0} \frac{1}{A} \int_A^A \lambda^k |m^{(k)}(\lambda)| \, d\lambda \leq M, \quad k = 1, \ldots, N,$$

for a constant $M$, then for every $p$, $1 < p < \infty$, the function $m$ is an $L^p$-multiplier for the operator $\pi(L)$.

**Proof.** Let us repeat, for the reader’s convenience, the same arguments as in the proof of Theorem 2.6 of [8] which allows us to assume that $G$ is stratified and that $L$ is a Rockland operator. Let $G$ be the nilpotent free group of the same nilpotency class as $G$ and let $X_1, \ldots, X_k$ be the free generators of the Lie algebra of $G$. Denote by $\alpha$ the homomorphism of $G$ onto $G$ sending $\exp X_j$ onto $\exp X_j$. Thus $\pi' = \pi \circ \alpha$ is a representation of $G$ induced by a unitary character of the normal connected subgroup $H = \alpha^{-1}(H)$ of $G$. Define the dilations $\delta_t$, $t > 0$, of free Lie algebras of $G$ by putting

$$\delta_t X_{-j} = t^{1/2n_j} X_j, \quad 1, \ldots, k.$$
Then

\[ L = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j} \]

is a Rockland operator on \( G \), \( \delta_j L = tL \), and \( \pi'(L) = \pi(L) \).

Proposition 2.2 of [8] asserts that \( G/H \), equipped with the Haar measure and the pseudometric \( \rho \) defined by

\[ \rho(x, y) = \inf \{|xy^{-1}z| : z \in H\}, \]

is a space of homogenous type. Denote by \( m^*_{G/H} \) the corresponding Hardy-Littlewood maximal function.

Consequently, Theorem 2.6 of [8] applied to the function \( K(\lambda) = (1 - \lambda)^N \) for \( \lambda \leq 1 \) and \( K(\lambda) = 0 \) for \( \lambda > 1 \), for sufficiently large \( N \), allows us to obtain an estimate

\[
\sup_{R > 0} \left| \int_0^R \left( 1 - \frac{\lambda}{K} \right)^N dE_{\pi(L)}(\lambda) f \right| \leq C \cdot m^*_{G/H}(f).
\]

Now, it remains to note that \( \pi(L) \) generates a semigroup of contractions on all \( L^p(G/H) \), \( 1 \leq p \leq \infty \). Denote by \( E_L(\lambda) \) the spectral resolution of \( L \) in \( L^2(G) \). By [5]

\[
\int_0^\infty \exp(-\lambda t) \, dE_L(\lambda) f = k_1 f, \quad f \in L^2(G),
\]

where \( k_1 \in L^1(G), \|k_1\|_1 = 1 \). Note that \( \{\pi(k_1)\}_{t > 0} \) is a semigroup on \( L^2(G/H) \) and that, for \( f \in L^2(G/H) \) and \( \varphi \in C_c(G) \),

\[
\lim_{t \to 0} t^{-1} \{ \pi(k_1) \pi(\varphi) f - \pi(\varphi) f \} = \pi(L) f = \pi(L) \pi(\varphi) f.
\]

Thus, \( \pi(L) \) is the infinitesimal generator for \( \{\pi(k_1)\}_{t > 0} \), which gives

\[
\int_0^\infty \exp(-\lambda t) \, dE_{\pi(L)}(\lambda) f = \pi(k_1) f, \quad f \in L^2(G/H).
\]

The operators \( \pi(x) \) act on \( L^2(G/H) \) by

\[
\pi(x) f(y) = a(x, y) f(y x),
\]

where the scalar function \( a \) is such that \( |a(x, y)| = 1 \). Thus \( \pi(x) \) is a contraction on every \( L^p(G/H) \) and so the same is true for the operator

\[
\pi(k_1) = \int_G \pi(x) k_1(x) \, dx.
\]

This completes the proof of the proposition.

Now, to see that Theorem 1 is a consequence of Proposition 2 we note that the Hermite operator \(-d^2/dx^2 + x^2\) is of the form \( \pi(L) \), where \( L \) is the sub-Laplacian on the Heisenberg group, and \( \pi \) is the Schrödinger representation on it.
Finally we mention also that in virtue of the recent results of W. Cupała [3], following the ideas of Hulanicki and Jenkins [8], our Proposition 2 produces an $L^p$-multiplier theorem for the eigenfunction expansions of the operators of the form

$$(-1)^k \frac{d^{2k}}{dx^{2k}} + p(x),$$

where $p(x)$ is a positive polynomial and $k$ is an arbitrary positive integer.

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REFERENCES


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