CONFORMAL MAPPINGS OF DOMAINS SATISFYING A WEDGE CONDITION

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Abstract. A plane Jordan curve $\Gamma$ satisfies an interior (exterior) wedge condition if for some $\alpha \in (0, 1)$ there is a fixed wedge of opening $\alpha \pi$ such that for any $\omega \in \Gamma$ one may place a wedge inside (outside) $\Gamma$ with vertex at $\omega$. Let $f$ be a conformal mapping of the disk $D$ onto the interior of $\Gamma$. We establish Hölder continuity of $f(f^{-1})$ on $\partial D(\Gamma)$ with best possible exponents in terms of $\alpha$.

1. Introduction. Let $\Gamma$ be a closed Jordan curve in the complex $\omega$-plane with interior $\Omega$ and exterior $\Omega^*$. We shall say that $\Gamma$ satisfies an interior $\alpha$-wedge condition if there exist $r > 0$ and $\alpha \in (0, 1)$ such that, for every $\omega \in \Gamma$, a closed circular sector of radius $r$ and opening $\alpha \pi$ lies in $\Omega$, with vertex at $\omega$. We say that $\Gamma$ satisfies an exterior $\alpha$-wedge condition if, for each $\omega \in \Gamma$ such a wedge lies in $\Omega^*$ with vertex at $\omega$. The interior wedge condition is often encountered in the study of partial differential equations where, together with its higher-dimensional analogs, it is called a “cone condition” [1, p. 233].

Let $f$ be a conformal mapping of $D = \{ \zeta : |\zeta| < 1 \}$ onto the interior of $\Gamma$. Our purpose here is to deduce Hölder continuity of $f$ or $f^{-1}$ on $\overline{D}$ or on $\overline{\Omega}$, respectively, from the appropriate $\alpha$-wedge condition. The interior $\alpha$-wedge condition is a special case of the one-sided smoothness condition studied by Pommerenke in [4] where, among other results, Hölder continuity of $f$ on $\overline{D}$ is established. The Hölder exponent obtained in [4] is not sharp in terms of $\alpha$, however. We shall obtain the best exponents for both $f$ and $f^{-1}$.

Theorem 1. Suppose that $\Gamma$ satisfies an interior $\alpha$-wedge condition. Then $f$ is Hölder continuous on $\overline{D}$ with exponent $\alpha$.

Theorem 2. Suppose that $\Gamma$ satisfies an exterior $\alpha$-wedge condition. Then $f^{-1}$ is Hölder continuous on $\overline{\Omega}$ with exponent $1/(2 - \alpha)$.

That the exponents are best possible is seen by taking $\Gamma$ to be a polygon.

I would like to thank Professor Dieter Gaier for suggesting this problem, and Professor S. E. Warschawski for several fruitful discussions.

Received by the editors February 22, 1984 and, in revised form, May 29, 1984.
1980 Mathematics Subject Classification. Primary 30C20.
Key words and phrases. Boundary behavior, Hölder continuity, cone condition.

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0002-9939/85 $1.00 + .25 per page

483
2. Geometric lemmas and reduction to strips. In this section we shall derive some purely geometric properties of curves satisfying an \( \alpha \)-wedge condition. For two points \( \omega_1, \omega_2 \in \Gamma \) we define \( C(\omega_1, \omega_2) \) to be the arc of \( \Gamma \) of smaller diameter between \( \omega_1 \) and \( \omega_2 \). The interior distance between \( \omega_1 \) and \( \omega_2 \) on \( \Gamma \), relative to \( \Omega \), is
\[
d_{\Omega}(\omega_1, \omega_2) = \inf \gamma \text{ diam } \gamma,
\]
where \( \gamma \) runs over all arcs from \( \omega_1 \) to \( \omega_2 \) which lie in \( \Omega \), except for their endpoints.

**Lemma 1 (Pommerenke [4, Theorem 1]).** Suppose that \( \Gamma \) satisfies an interior \( \alpha \)-wedge condition. Then there exists a constant \( M_1 > 0 \), depending only on \( \Gamma \), such that for \( \omega_1, \omega_2 \in \Gamma \)
\[
diam C(\omega_1, \omega_2) \leq M_1 d_{\Omega}(\omega_1, \omega_2).
\]

For \( \omega_0 \in \Gamma \) and \( \rho > 0 \), each component of \( \{ \omega : |\omega - \omega_0| = \rho \} \cap \Omega \) is a crosscut of \( \Omega \). If \( \rho < r \), then exactly one of these crosscuts intersects the \( \alpha \)-wedge at \( \omega_0 \). We shall call this the \( \rho \) crosscut of \( \Omega \) at \( \omega_0 \).

**Lemma 2.** Suppose that \( \Gamma \) satisfies an interior \( \alpha \)-wedge condition and that \( 0 \in \Gamma \). Let \( 0, \omega', \omega^* \) be consecutive points on \( C(0, \omega^*) \) with \( \rho = |\omega^*| < r \), and \( \omega^* \) on the \( \rho \) crosscut of \( \Omega \) at \( 0 \). Then there exists a constant \( M_2 > 0 \), depending on \( \Gamma \) such that
\[
|\omega'| < M_2.
\]

**Proof.** By Lemma 1,
\[
|\omega'| \leq diam C(0, \omega^*) \leq M_1 d_{\Omega}(0, \omega^*) < M_1 (|\omega^*| + 2\pi|\omega^*|) = M_2|\omega^*|,
\]
since there is a path \( \gamma \) from \( 0 \) to \( \omega^* \) in \( \Omega \) which consists of a ray inside the \( \alpha \) wedge at \( 0 \) and a subarc of the \( \rho \) crosscut at \( 0 \).

**Lemma 3.** Let \( \Gamma \) satisfy an interior \( \alpha \)-wedge condition and let \( 0, \omega_1, \omega^*, \omega_2 \) be consecutive points on \( \Gamma \) with \( \omega^* \in C(\omega_1, \omega_2), |\omega_1| = |\omega_2| = \rho \), and such that \( \omega_1 \) and \( \omega_2 \) are endpoints of an arc of the circle \( \{ |\omega| = \rho \} \) lying in \( \Omega \). Then there exists a constant \( M_3 > 0 \) depending on \( \Gamma \) for which
\[
|\omega^*|/\rho < M_3.
\]

**Proof.** Using Lemma 1 we have for \( |\omega^*| > \rho \),
\[
|\omega^*| - |\omega_1| \leq |\omega^* - \omega_1| \leq diam C(\omega_1, \omega_2) \leq M_1 d_{\Omega}(\omega_1, \omega_2) < 2\pi\rho M_1.
\]
Thus \( |\omega^*|/\rho < 2\pi M_1 + 1 = M_3 \).

Of course, Lemma 3 (and the others) apply with the words "exterior" and "\( \Omega^* \)" replacing "interior" and "\( \Omega \)". In this form it will be used to prove Lemma 4, which assumes the exterior condition. It is convenient to state and prove Lemma 4 in the context to which it will be applied in the proof of Theorem 2, so we shall now transform \( \Omega \) and \( \Omega^* \) into strip domains.
Suppose that 0 and $\omega_0$ are on $\Gamma$ and that $|\omega_0| = (\text{diam } \Gamma)/2 = R > 2r$. Suppose also that $r$ is so small that $|\omega_1 - \omega_2| < 2r$ implies that diam $C(\omega_1, \omega_2) < R/2$. For a particular $\omega' \in \Gamma$, with $|\omega'| < r$, let $\Gamma_1$ be the open arc of $\Gamma$ between 0 and $\omega_0$ which contains $\omega'$ and let $\Gamma_2 = \Gamma - \Gamma_1$. Next consider the mapping

$$w(\omega) = \log((\omega - \omega_0)/\omega)$$

which may be defined in $\Omega$ and $\Omega^*$ so that

1. $w(\Omega)$ is an infinite strip $S$ in the $w$-plane, bounded by curves $C_1 = w(\Gamma_1)$ and $C_2 = w(\Gamma_2)$ with $-\infty$ and $+\infty$ as boundary points;
2. $w(\Omega^*)$ is a strip $S^*$ bounded by curves $C_1$ and $C_2'$, where we may assume that $C_2' = \{w - 2\pi i, w \in C_2\}$.

Now, let $L$ be a piecewise analytic arc from $-\infty$ to $+\infty$ in $S$, and for $u$ real, let $\Lambda_u = \{w: \text{Re } w = u\}$. We define $\sigma(u)$ to be the maximal closed subarc of $\Lambda_u \cap S$ which is the first (moving along $L$ from $-\infty$ to $+\infty$) to be crossed by $L$ an odd number of times.

**Lemma 4.** Suppose that $\Gamma$ satisfies an exterior $\alpha$-wedge condition and that $S$ and $S^*$ are strip domains corresponding to $\Omega$ and $\Omega^*$, as above. Suppose that $w^* = u^* + iv^* \in \sigma(u^*) \cap C_1$ and that $w^*$ separates $w' = u' + iv' \in C_1$ from $+\infty$, with $u' > \log((R-r)/r)$.

Then there exists a constant $M_4' > 0$ depending only on $\Gamma$ such that

$$u' - u^* < M_4'. \tag{2.1}$$

**Remark.** In the $\omega$-plane, this means that for $\omega^*$ on the circular crosscut of $\Omega$ at 0 corresponding to $\sigma(u^*)$, with $\omega^* \in C(0, \omega')$ and $|\omega'| < r$, we have

$$\left|\frac{\omega^*}{\omega'}\right| < M_4, \quad \text{where } M_4 = \frac{R + r}{R - (R/2)} e^{M_4'} \tag{2.2}.$$

**Proof.** We first observe that $C_1$ and $C_2'$ are "below" $L$ in the sense that they are in the component of $C - L$ with $-i\infty$ as a boundary point. Similarly $C_2$ is "above" $L$.

![Figure 1](https://www.ams.org/journal-terms-of-use)
Now, let $\gamma_1$ be a piecewise analytic arc in $S^*$ from $w^*$ to $+\infty$ (see Figure 1). We may assume that $\sigma(u^*)$ intersects $L$ at exactly one point and we let $\gamma_2$ be the segment of $\sigma(u^*)$ between $w^*$ and that point. Then let $\gamma_3$ be the subarc of $L$ from that point of intersection to $+\infty$. Then $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is the boundary of a half strip domain $A$. Let $C_1^+$ be the open subarc of $C_1$ from $w^*$ to $+\infty$ and let $C_1^-$ be the open subarc from $w^*$ to $-\infty$. Then $C_1^+ \subset A$ and $C_1^- \subset \bar{C} - A$.

Next, pick $w'' \in C_1^+ \cap \Lambda_u^*$, and consider the segment $s = w'''w'$. Then $s$ must cross $\gamma_1 \cup \gamma_2 \cup \gamma_3$ an odd number of times. It does not cross $\gamma_2$ at all and it cannot cross $\gamma_3$ an odd number of times, since $w'$ and $w''$ are both below $L$. Thus some component $l$ of $s \cap S^*$ must cross $\gamma_1$ an odd number of times, so that one endpoint, call it $w_1$, is on $C_1^-$. The other endpoint $w_2$ must be on $\partial S^*$, so it is not on $C_2$. Furthermore, the exterior wedge condition for $\Gamma$ implies that $S^*$ contains a parallel strip separating $C_1$ and $C_2$ so that $s$ is above the strip and $C_2$ is below it. Thus $w_2 \notin C_2$, and we conclude that $w_2 \in C_1^-$. But then Lemma 4 follows from Lemma 3 applied to $\beta^*$: Let $\omega_1$, $\omega_2$, $\omega_*$ satisfy the hypotheses of Lemma 3, so that (2.2) and (2.1) follow since $|\omega| = |\omega_1|$.

3. Hölder continuity of the mapping functions.

**Proof of Theorem 1.** We now assume that $f$ maps $D$ conformally onto $\Omega$ and $\bar{D}$ homeomorphically onto $\overline{\Omega}$. We must show that there exist positive $\delta$ and $K$ such that, for any $\xi_0 \in \partial D$ and $\xi \in \partial D$ with $|\xi - \xi_0| < \delta$, we have

$$|f(\xi) - f(\xi_0)| < K|\xi - \xi_0|^{\alpha}.$$ 

We assume that $f(\xi_0) = 0$ and that $f(-\xi_0) = \omega_0$ where $|\omega_0| = (\text{diam } \Gamma)/2 = R > 2r$ and $r$ is so small that $|\omega_1 - \omega_2| < 2r$ implies that $\text{diam } C(\omega_1, \omega_2) < R/2$. Now let $z = \log((\xi_0 + \xi)/(\xi_0 - \xi))$ and $w = \log(1/\omega)$, for $\omega \in \Omega$. It then suffices to consider the mapping $w(z) = u(z) + iv(z)$ from $\Sigma = \{x + iy: |y| < \pi/2\}$ onto the half strip $S$ which is the image of $\Omega$ under the logarithmic mapping. We then show that, for $x_1$ fixed and depending only on $f$ and $x_2 > x_1$ with $z_2 = x_2 + iy_2 \in \partial \Sigma$, we have

$$(3.1) \quad ax_2 - u(z_2) \leq M,$$

where $M$ depends only on $f$. As in [3] we prove (3.1) by comparing modules of quadrilaterals.

On account of the $\alpha$-wedge condition at $0 \in \Gamma$, the strip $S$ contains a parallel half strip $S'$ of width $\alpha r$. We may assume that $S' \supset \{w = u: u \geq u_0 = \log r\}$. For $u \geq u_0$, let $\sigma(u)$ be the maximal subarc of $\Lambda_u \cap \hat{S}$ which intersects $S'$. Now choose $x_1 = \max\{\text{Re } z(w), w \in \sigma(u_0)\}$ so that $x_1$ depends only on $f$. For $x \geq x_1$, define

$$\gamma_x = \{w(z): \text{Re } z = x, |y| \leq \pi/2\},$$

$$\bar{u}(x) = \sup\{u: \sigma(u) \cap \gamma_x \neq \emptyset\},$$

and

$$u(x) = \inf\{u: \sigma(u) \cap \gamma_x \neq \emptyset\}.$$
For \( x_2 > x_1 \), let \( Q = \{ w(z): z = x + iy \in \Sigma, \ x_1 < x < x_2 \} \), \( Q' = \{ w \in S': \ u(x_1) < u < \tilde{u}(x_2) \} \) and \( Q = \) the component of \( \hat{S} - \sigma(\tilde{u}(x_2)) - \sigma(u(x_1)) \) which contains \( Q' \) (see Figure 2). \( Q' \) is a rectangle, while \( Q \) and \( \tilde{Q} \) are quadrilaterals, two of whose sides are "horizontal" subarcs of \( C_1 \) and \( C_2 \). Let \( M(Q) \) be the module of the family of curves connecting the horizontal sides of \( Q \) with corresponding definitions for \( M(Q') \) and \( M(\tilde{Q}) \). Then by conformal invariance and the comparison principle for modules,

\[
\frac{1}{\alpha} (x_2 - x_1) = M(Q) \leq M(\tilde{Q}) \leq M(Q') = \frac{\tilde{u}(x_2) - u(x_1)}{\alpha \pi}
\]

so that

\[
\alpha x_2 - \tilde{u}(x_2) \leq \alpha x_1 - u(x_1) \leq K_1
\]

where \( K_1 \) depends on \( f \). Now suppose that \( z_2 = x_2 - i\pi/2 \) and that \( w(z_2) \in C_1 \). Then

\[
\alpha x_2 - u(z_2) \leq K_1 + \tilde{u}(x_2) - u(x_2) + u(x_2) - u(z_2).
\]

Since the width of \( S \) is at most \( 2\pi \), one may use the Ahlfors distortion theorem (see [3, p. 317]) to show that

\[
\tilde{u}(x_2) - u(x_2) \leq 4\pi.
\]

It remains only to show that \( u(x_2) - u(z_2) \) is bounded. To see this let \( w^* = u^* + iv^* \in \sigma(u(x_2)) \cap C_1 \). Let \( C_1^+ \) be the closed subarc of \( C_1 \) from \( w^* \) to \( +\infty \). Let \( u' = \min\{ u: w = u + iv \in C_1^+ \} \). Let \( w' = u' + iv \in C_1^+ \). Then \( \gamma_{x_2} \) is in the component of \( \hat{S} - \sigma(u(x_2)) \) with \( +\infty \) as a boundary point so that \( \gamma_{x_2} \cap C_1 \in C_1^+ \), and \( u(z_2) \geq u' \). But then \( w' \) separates \( w^* \) from \( +\infty \), and we may apply Lemma 2 to see that

\[
u(x_2) - u(z_2) \leq u^* - u' < \log M_2
\]

and Theorem 1 is proved.
Proof of Theorem 2. We now suppose that $\Gamma$ satisfies an exterior $\alpha$-wedge condition and apply Theorem 1 to the exterior mapping $f^* : \{|f| > 1\} = D^* \rightarrow \Omega^*$ to see that $f^*$ is Hölder continuous on $D^*$. We then apply Theorem 1 of [2] to see that $f^{-1}$ is Hölder continuous with exponent $1/(2 - \alpha)$ for approach in the "kernel". That is, considering without loss of generality $0 \in \Gamma$, there exists $K > 0$ such that

$$|f^{-1}(\omega) - f^{-1}(0)| < K|\omega|^{1/(2 - \alpha)}$$

for $\omega$ on the crosscut of $\Omega$ at 0 as in the Remark to Lemma 4. We will establish (3.2) for any $\omega \in \Gamma$ with $|\omega| < \delta$, for some $\delta > 0$. We now let

$$z = \log((\xi_0 + \xi)/(\xi_0 - \xi)) \quad \text{and} \quad w = \log((\omega - \omega_0)/\omega)$$

and consider the mapping $z(w) = x(w) + iy(w)$ from $S$ onto $\Sigma$ where $S$ is the infinite strip as in the proof of Lemma 4. Theorem 1 of [2] guarantees that for $w \in \sigma(u)$,

$$(3.3) \quad \frac{1}{(2 - \alpha)}u - x(w) \leq M$$

for $M$ depending only on $f$. We shall show that for any $w \in C_1$ with $u = \Re w > \log ((R - r)/r) = u_0$, (3.3) holds with $M$ replaced by $M'$, which again depends only on $f$. So, let $u > u_0$ be given and $w = u + iv \in C_1$. Let $u^* = u - M'_4$ for $M'_4$ as in (2.1). It follows from Lemma 4 that $w^* = \sigma(u^*) \cap C_1$ is separated from $+\infty$ by $w$. Then

$$\frac{1}{(2 - \alpha)}u - x(w) \leq \frac{1}{(2 - \alpha)}u^* - x(w^*) + \frac{1}{(2 - \alpha)}(u - u^*) + x(w^*) - x(w)$$

$$\leq M + \frac{1}{(2 - \alpha)}M'_4 = M'.$$

References


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