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ISOMETRIES OF $L^1 \cap L^p$

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ABSTRACT. Denote by $L^1 \cap L^p$ the Banach space of all functions f such that $f \in L^1$ and $f \in L^p$ with norm $|||f||| = \max(\|f\|_1, \|f\|_p)$. We give a characterization of isometries of $L^1 \cap L^p$ and show that T is an isometry if and only if T is of the form $Tf = h\Phi(f)$, where Φ is an operator generated by a regular set isomorphism and h is a suitable function.

In his monograph [**1**] S. Banach gave the characterization of L^p-isometries. This result was generalized by J. Lamperti [**4**]. The isometries of Orlicz spaces were determined by G. Lumer [**5, 6**]. For generalizations in various directions see [**8, 7, 2, 3**].

In this paper we give a characterization of the isometries of $L^1 \cap L^p$, $1 < p < \infty$. Our result is valid for either the real or the complex cases.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and, as usual, denote by $L^p(X)$, $1 \leqslant p \leqslant \infty$, the class of all measurable functions whose pth power is integrable over X. $L^1(X) \cap L^p(X)$ denotes the Banach space consisting of all f such that $f \in L^1(X)$ and $f \in L^p(X)$ with norm

$$|||f||| = \max(\|f\|_1, \|f\|_p).$$

The space $L^1(X) \cap L^p(X)$ is a rearrangement-invariant function space. It is isomorphic (not isometric) to the Orlicz space $L^\varphi(X)$ with norm

$$\|f\|_\varphi = \inf\left\{\alpha > 0 : \int \varphi\left(\left|\frac{f}{\alpha}\right|\right) d\mu \leqslant 1\right\},$$

where

$$\varphi(t) = \begin{cases} t & \text{for } t \leqslant 1, \\ t^p & \text{for } t > 1. \end{cases}$$

It should be pointed out that if $\mu(X) < \infty$, then $\|f\|_1 \leqslant [\mu(X)]^{1-1/p}\|f\|_p$ for $f \in L^1(X) \cap L^p(X)$. In particular, $|||f||| = \|f\|_p$ if $\mu(X) \leqslant 1$. Therefore an isometry of $L^1(X) \cap L^p(X)$ is the L^p-isometry and vice versa in the case $\mu(X) \leqslant 1$. If $(X, \mathscr{F}, \mu)$ has only atoms with a mass greater than or equal to one, then $\|f\|_p \leqslant \|f\|_1$. Thus, in particular, $l^1 \cap l^p$ and l^1 are isometrically isomorphic.

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A set map $\phi: \mathscr{F} \to \mathscr{F}$ (defined modulo zero measure sets) is called a regular set isomorphism if:

(i) $\phi(X \setminus A) = \phi(X) \setminus \phi(A)$, $A \in \mathscr{F}$;

(ii) $\phi(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \phi(A_n)$ for disjoint $A_n \in \mathscr{F}$;

(iii) $\mu(\phi(A)) = 0$ if and only if $\mu(A) = 0$, $A \in \mathscr{F}$.

A regular set isomorphism induces a positive linear operator, also denoted by ϕ, from the set of (equivalence classes of) measurable functions into itself defined by $1_A \to 1_{\phi(A)}$. Clearly, $\phi(A) \cap \phi(B) = \varnothing$ if and only if $A \cap B = \varnothing$. Let $\operatorname{supp} f = \{x: f(x) \neq 0\}$ for any function f. All statements about sets and functions are modulo sets of μ-measure zero.

Note that the description of isometries in Lamperti's Theorem 3.1 [**4**] is not quite correct (cf. [**2**]). Actually, Lamperti has proved that an L^p-isometry is of the form

$$(Tf)(x) = h(x)(\phi(f))(x),$$

where ϕ is a regular set isomorphism and $E\{|h|^p|\phi(\mathscr{F})\} = d(\mu \circ \phi^{-1})/d\mu$, i.e.

$$\int_{\phi(A)} |h|^p \, d\mu = \int_{\phi(A)} \frac{d(\mu \circ \phi^{-1})}{d\mu} \, d\mu \quad \text{for all } A \in \mathscr{F}.$$

DEFINITION. We say that the σ-field $\mathscr{F}$ has property EW if there exist $A \in \mathscr{F}$, with $\mu(A) < 1$, and a family of sets $\mathscr{G} \subset \mathscr{F}$ with the following properties: (1) for each disjoint $A, B \in \mathscr{G}$ there exist $C, D \in \mathscr{F}$ such that A, B, C, D are disjoint and $1 \leqslant \mu(C) < \infty$, $1 \leqslant \mu(D) < \infty$; (2) $\{1_A: A \in \mathscr{G}\}$ is total in $L^1(X)$.

For instance, if $2 < a \leqslant \infty$, then the interval $(0, a)$, with Lebesgue measurable subsets as a σ-field and Lebesgue measure, has property EW.

THEOREM. *Let $\mathscr{F}$ have property* EW, *and let T be an isometry of $L^1(X) \cap L^p(X)$, $1 < p < \infty$. Then there exists a regular set isomorphism ϕ and a function h, with*

$$E\{|h| \, |\phi(\mathscr{F})\} = E\{|h|^p \, |\phi(\mathscr{F})\} = d(\mu \circ \phi^{-1})/d\mu,$$

such that

$$(Tf)(x) = h(x)(\phi(f))(x).$$

We use the following fact.

LEMMA. *Let $g_k \in L^1(X)$, $k = 1, 2, 3, 4$. If*

$$\|\varepsilon_1 g_1 + \varepsilon_2 g_2 + \varepsilon_3 g_3 + \varepsilon_4 g_4\|_1 = \|\varepsilon_1 g_1 + \varepsilon_3 g_3\|_1 + \|\varepsilon_2 g_2 + \varepsilon_4 g_4\|_1$$

for all $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ such that $\varepsilon_k = 1$ or -1, $k = 1, 2, 3, 4$, then $\operatorname{supp} g_1$ *and* $\operatorname{supp} g_2$ *are disjoint.*

PROOF. If $\|h_1 \pm h_2\|_1 = \|h_1\|_1 + \|h_2\|_1$ for $h_1, h_1 \in L^1(X)$, then $h_1 h_2 = 0$. Therefore, by assumption, we have

$$(g_1 + g_3)(g_2 + g_4) = 0, \quad (g_1 - g_3)(g_2 + g_4) = 0,$$
$$(g_1 + g_3)(g_2 - g_4) = 0, \quad (g_1 - g_3)(g_2 - g_4) = 0.$$

Summing the equalities above, we obtain $4 g_1 g_2 = 0$.

PROOF OF THE THEOREM. Observe that for $1_A \in L^1(X) \cap L^p(X)$ we have

$$|||1_A||| = \begin{cases} \mu(A) & \text{if } \mu(A) \geqslant 1, \\ \sqrt[p]{\mu(A)} & \text{if } \mu(A) \leqslant 1. \end{cases}$$

Let $A, B \in \mathscr{F}$ be disjoint with $1 \leqslant \mu(A) < \infty$ and $1 \leqslant \mu(B) < \infty$. Obviously, $\|T1_A\|_p \leqslant |||1_A||| = \mu(A)$ and $\|T1_B\|_p \leqslant \mu(B)$. Suppose $\|T1_{A \cup B}\|_p = \mu(A \cup B)$. Then

$$\mu(A) + \mu(B) = \|T1_{A \cup B}\|_p \leqslant \|T1_A\|_p + \|T1_B\|_p \leqslant \mu(A) + \mu(B).$$

By the strict convexity of the p-norm we obtain $T1_A/\mu(A) = T1_B/\mu(B)$. But this is impossible, since then we would have

$$|||T(1_A/\mu(A) - 1_B/\mu(B))||| = 0 \neq |||1_A/\mu(A) - 1_B/\mu(B)|||.$$

This contradiction shows that $\|T1_{A \cup B}\|_1 = \mu(A \cup B)$, $\|T1_A\|_1 = \mu(A)$, and $\|T1_B\|_1 = \mu(B)$. A similar argument shows that $\|Th\|_1 = \mu(A)$ if $|h| = 1_A$ and $\mu(A) \geqslant 1$.

Now let $A, B \in \mathscr{G}$ be disjoint sets. Let $C, D \in \mathscr{F}$ be such that $1 \leqslant \mu(C) < \infty$, $1 \leqslant \mu(D) < \infty$, and the sets A, B, C, D are disjoint. Put $\varepsilon_k = 1$ or -1 for $k = 1, 2, 3, 4$. We have

$$\begin{aligned} \|T(\varepsilon_1 1_A + \varepsilon_2 1_B + \varepsilon_3 1_C + \varepsilon_4 1_D)\|_1 &= \mu(A \cup B \cup C \cup D) \\ &= \|T(\varepsilon_1 1_A + \varepsilon_3 1_C)\|_1 + \|T(\varepsilon_2 1_B + \varepsilon_4 1_D)\|_1. \end{aligned}$$

By the Lemma $\operatorname{supp} T1_A \cap \operatorname{supp} T1_B = \varnothing$. Obviously, $\operatorname{supp} T1_A \cap \operatorname{supp} T1_D = \varnothing$ also. Hence,

$$\mu(A \cup D) = \|T(1_A + 1_D)\|_1 = \|T1_A\|_1 + \|T1_D\|_1 = \|T1_A\|_1 + \mu(D),$$

so $\|T1_A\|_1 = \|1_A\|_1$.

This shows T is an L^1-isometry on $L^1(X) \cap L^p(X)$, since $\{1_A \colon A \in \mathscr{G}\}$ is total and T maps disjointly supported indicators onto disjointly supported functions. Further, T extends by continuity to an L^1-isometry on all of $L^1(X)$.

Now let $A, B \in \mathscr{F}$ be disjoint such that $\mu(A) < 1$ and $\mu(B) < \infty$. Then $|||1_A||| > \|1_A\|_1 = \|T1_A\|_1$. Thus,

$$\|1_A\|_p = |||1_A||| = |||T1_A||| = \|T1_A\|_p.$$

Put $\varepsilon > 0$ so small that

$$\|T(1_A + \varepsilon 1_B)\|_p > \|T(1_A + \varepsilon 1_B)\|_1 \quad \text{and} \quad \|1_A + \varepsilon 1_B\|_p > \|1_A + \varepsilon 1_B\|_1.$$

We have

$$\begin{aligned} \|T1_A\|_p^p + \varepsilon^p \|T1_B\|_p^p &= \|T(1_A + \varepsilon 1_B)\|_p^p = |||T(1_A + \varepsilon 1_B)|||^p = |||1_A + \varepsilon 1_B|||^p \\ &= \|1_A + \varepsilon 1_B\|_p^p = \|1_A\|_p^p + \varepsilon^p \|1_B\|_p^p, \end{aligned}$$

so $\|T1_B\|_p = \|1_B\|_p$. Therefore, T is also an L^p-isometry.

From this and the correct version of Lamperti's Theorem 3.1 [**4**] (in the case $p = 1$),

$$Tf = h\phi(f),$$

where ϕ is an operator generated by a regular set isomorphism. We have

$$E\{|h|\,|\phi(\mathscr{F})\} = \frac{d(\mu \circ \phi^{-1})}{d\mu} = E\{|h|^p\,|\phi(\mathscr{F})\}$$

because T is an L^1- and L^p-isometry. It should be pointed out that these equalities can also be written for $p = 2$ since T maps functions with disjoint supports to functions with disjoint supports.

EXAMPLE. $E\{|H|\,|\phi(\mathscr{F})\} = E\{|h|^p|\phi(\mathscr{F})\} \not\Rightarrow |h| = |h|^p$. The spaces $L^1(0,\infty) \cap L^p(0,\infty)$ and $L^1(-\infty,\infty) \cap L^p(-\infty,\infty)$ are isometrically isomorphic because there exists an invertible measure-preserving transformation τ: $(-\infty,\infty) \to (0,\infty)$ (e.g., $\tau(x) = |x + [x]|$). Thus, consider now an operator T: $L^1(0,\infty) \cap L^p(0,\infty) \to L^1(-\infty,\infty) \cap L^p(-\infty,\infty)$ defined by

$$(Tf)(x) = \begin{cases} af(-x/\alpha) & \text{for } x < 0, \\ bf(x/\beta) & \text{for } x \geqslant 0, \end{cases}$$

where α, β, a, b are positive and satisfy $a\alpha + b\beta = a^p\alpha + b^p\beta = 1$. Obviously,

$$Tf = h\phi(f), \quad \text{where } h(x) = \begin{cases} a & \text{for } x < 0, \\ b & \text{for } x \geqslant 0, \end{cases}$$

$\phi(A) = (-\alpha A) \cup (\beta A)$ $(\gamma A = \{\gamma x : x \in A\}, \gamma \in \mathbf{R})$. We have

$$E\{|h|^r\,|\phi(\mathscr{F})\} = \frac{a^r\alpha + b^r\beta}{\alpha + \beta}.$$

Now let $1 \leqslant p_1 < p_2 < \infty$. If we choose $\alpha, \beta, a, b > 0$ such that $a^{p_1}\alpha + b^{p_1}\beta = a^{p_2}\alpha + b^{p_2}\beta = 1$, then T, defined as above, is an L^{p_1}- and L^{p_2}-isometry, but not necessarily an isometry for other $p \in [1, \infty)$. This example breaks down Corollary 3.1 in [**4**]. It should be noted that if $1 \leqslant p_1 < p_2 < \infty$ and $0 < a < 1 < b$ (or $a > 1 > b > 0$) then the solutions to our equations give us positive α, β.

A function τ: $X \to X$ is called an invertible measure-preserving transformation if τ^{-1} exists, τ and τ^{-1} are measurable, and $\mu(\tau(A)) = \mu(A)$ for all $A \in \mathscr{F}$.

COROLLARY. *Let T be a surjective isometry of $L^1(0,\infty) \cap L^p(0,\infty)$, $1 < p < \infty$. Then there exists an invertible measure-preserving transformation τ: $(0,\infty) \to (0,\infty)$ and a measurable function r with $|r| = 1$ such that*

$$(Tf)(x) = r(x)r(\tau(x)).$$

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