ISOMETRIES OF $L^1 \cap L^p$

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Abstract. Denote by $L^1 \cap L^p$ the Banach space of all functions $f$ such that $f \in L^1$ and $f \in L^p$ with norm $\|f\| = \max(\|f\|_1, \|f\|_p)$. We give a characterization of isometries of $L^1 \cap L^p$ and show that $T$ is an isometry if and only if $T$ is of the form $Tf = h\Phi(f)$, where $\Phi$ is an operator generated by a regular set isomorphism and $h$ is a suitable function.

In his monograph [1] S. Banach gave the characterization of $L^p$-isometries. This result was generalized by J. Lamperti [4]. The isometries of Orlicz spaces were determined by G. Lumer [5, 6]. For generalizations in various directions see [8, 7, 2, 3].

In this paper we give a characterization of the isometries of $L^1 \cap L^p$, $1 < p < \infty$. Our result is valid for either the real or the complex cases.

Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and, as usual, denote by $L^p(X)$, $1 \leq p \leq \infty$, the class of all measurable functions whose $p$th power is integrable over $X$. $L^1(X) \cap L^p(X)$ denotes the Banach space consisting of all $f$ such that $f \in L^1(X)$ and $f \in L^p(X)$ with norm

$$\|f\| = \max(\|f\|_1, \|f\|_p).$$

The space $L^1(X) \cap L^p(X)$ is a rearrangement-invariant function space. It is isomorphic (not isometric) to the Orlicz space $L^\varphi(X)$ with norm

$$\|f\|_\varphi = \inf \left\{ \alpha > 0 : \int \varphi\left(\frac{|f|}{\alpha}\right) d\mu \leq 1 \right\},$$

where

$$\varphi(t) = \begin{cases} t & \text{for } t \leq 1, \\ t^p & \text{for } t > 1. \end{cases}$$

It should be pointed out that if $\mu(X) < \infty$, then $\|f\|_1 \leq [\mu(X)]^{1-1/p}\|f\|_p$ for $f \in L^1(X) \cap L^p(X)$. In particular, $\|f\|_1 = \|f\|_p$ if $\mu(X) \leq 1$. Therefore an isometry of $L^1(X) \cap L^p(X)$ is the $L^p$-isometry and vice versa in the case $\mu(X) \leq 1$. If $(X, \mathcal{F}, \mu)$ has only atoms with a mass greater than or equal to one, then $\|f\|_p \leq \|f\|_1$. Thus, in particular, $l^1 \cap l^p$ and $l^1$ are isometrically isomorphic.
A set map $\phi : \mathcal{F} \to \mathcal{F}$ (defined modulo zero measure sets) is called a regular set isomorphism if:

(i) $\phi (X \setminus A) = \phi (X) \setminus \phi (A)$, $A \in \mathcal{F}$;
(ii) $\phi (\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \phi (A_n)$ for disjoint $A_n \in \mathcal{F}$;
(iii) $\mu (\phi (A)) = 0$ if and only if $\mu (A) = 0$, $A \in \mathcal{F}$.

A regular set isomorphism induces a positive linear operator, also denoted by $\phi$, from the set of (equivalence classes of) measurable functions into itself defined by $l_A \to l_{\phi (A)}$. Clearly, $\phi (A) \cap \phi (B) = \emptyset$ if and only if $A \cap B = \emptyset$. Let $\text{supp} \, f = \{ x : f(x) \neq 0 \}$ for any function $f$. All statements about sets and functions are modulo sets of $\mu$-measure zero.

Note that the description of isometries in Lamperti’s Theorem 3.1 [4] is not quite correct (cf. [2]). Actually, Lamperti has proved that an $L^p$-isometry is of the form

$$(Tf)(x) = h(x)(\phi (f))(x),$$

where $\phi$ is a regular set isomorphism and $E\{|h|^p |\phi (\mathcal{F})\} = d(\mu \circ \phi^{-1})/d\mu$, i.e.

$$\int_{\phi (A)} |h|^p \, d\mu = \int_{\phi (A)} d(\mu \circ \phi^{-1}) \, d\mu \text{ for all } A \in \mathcal{F}.$$ 

**Definition.** We say that the $\sigma$-field $\mathcal{F}$ has property EW if there exist $A \in \mathcal{F}$, with $\mu (A) < 1$, and a family of sets $\mathcal{G} \subset \mathcal{F}$ with the following properties: (1) for each disjoint $A, B \in \mathcal{F}$ there exist $C, D \in \mathcal{F}$ such that $A, B, C, D$ are disjoint and $1 < \mu (C) < \infty$, $1 < \mu (D) < \infty$; (2) $\{1_A : A \in \mathcal{G}\}$ is total in $L^1 (X)$.

For instance, if $2 < a < \infty$, then the interval $(0, a)$, with Lebesgue measurable subsets as a $\sigma$-field and Lebesgue measure, has property EW.

**Theorem.** Let $\mathcal{F}$ have property EW, and let $T$ be an isometry of $L^1 (X) \cap L^p (X)$, $1 < p < \infty$. Then there exists a regular set isomorphism $\phi$ and a function $h$, with

$$E\{|h| |\phi (\mathcal{F})\} = E\{|h|^p |\phi (\mathcal{F})\} = d(\mu \circ \phi^{-1})/d\mu,$$

such that

$$(Tf)(x) = h(x)(\phi (f))(x).$$

We use the following fact.

**Lemma.** Let $g_k \in L^1 (X)$, $k = 1, 2, 3, 4$. If

$$\|\epsilon_1 g_1 + \epsilon_2 g_2 + \epsilon_3 g_3 + \epsilon_4 g_4\|_1 = \|\epsilon_1 g_1 + \epsilon_3 g_3\|_1 + \|\epsilon_2 g_2 + \epsilon_4 g_4\|_1$$

for all $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ such that $\epsilon_k = 1$ or $-1$, $k = 1, 2, 3, 4$, then $\text{supp} \, g_1$ and $\text{supp} \, g_2$ are disjoint.

**Proof.** If $\|h_1 \pm h_2\|_1 = \|h_1\|_1 + \|h_2\|_1$ for $h_1, h_2 \in L^1 (X)$, then $h_1 h_2 = 0$. Therefore, by assumption, we have

$$(g_1 + g_3)(g_2 + g_4) = 0, \quad (g_1 - g_3)(g_2 + g_4) = 0,$$

$$(g_1 + g_3)(g_2 - g_4) = 0, \quad (g_1 - g_3)(g_2 - g_4) = 0.$$

Summing the equalities above, we obtain $4g_1 g_2 = 0$. 

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Proof of the Theorem. Observe that for \(1 \leq \mu(A) < \infty\) and \(1 \leq \mu(B) < \infty\). Obviously, 
\[
\|T_{1_A}\|_p \leq \|\mu(A)\|_p = \mu(A) \quad \text{and} \quad \|T_{1_B}\|_p \leq \mu(B).
\]
Suppose \(\|T_{1_A \cup B}\|_p = \mu(A \cup B)\). Then 
\[
\mu(A) + \mu(B) = \|T_{1_A \cup B}\|_p \leq \|T_{1_A}\|_p + \|T_{1_B}\|_p \leq \mu(A) + \mu(B).
\]
By the strict convexity of the \(p\)-norm we obtain 
\[
\|T_{1_A/\mu(A)} - 1_B/\mu(B)\|_p = 0 \neq \|1_A/\mu(A) - 1_B/\mu(B)\|_p.
\]
This contradiction shows that 
\[
\|T_{1_A \cup B}\|_1 = \mu(A \cup B), \quad \|T_{1_A}\|_1 = \mu(A), \quad \text{and} \quad \|T_{1_B}\|_1 = \mu(B).
\]
A similar argument shows that \(\|Th\|_1 = \mu(A)\) if \(|h| = 1_A\) and \(\mu(A) \geq 1\).

Now let \(A, B \in \mathcal{F}\) be disjoint sets. Let \(C, D \in \mathcal{F}\) be such that \(1 \leq \mu(C) < \infty\), \(1 \leq \mu(D) < \infty\), and the sets \(A, B, C, D\) are disjoint. Put \(\epsilon_k = 1\) or \(-1\) for \(k = 1, 2, 3, 4\). We have
\[
\|T(\epsilon_11_A + \epsilon_21_B + \epsilon_31_C + \epsilon_41_D)\|_1 = \|T(\epsilon_11_A + \epsilon_31_C)\|_1 + \|T(\epsilon_21_B + \epsilon_41_D)\|_1.
\]
By the Lemma \(\text{supp } T_{1_A} \cap \text{supp } T_{1_B} = \emptyset\). Obviously, \(\text{supp } T_{1_A} \cap \text{supp } T_{1_D} = \emptyset\) also. Hence,
\[
\mu(A \cup D) = \|T(1_A + 1_D)\|_1 = \|T_{1_A}\|_1 + \|T_{1_D}\|_1 = \|T_{1_A}\|_1 + \mu(D),
\]
so \(\|T_{1_A}\|_1 = \|1_A\|_1\).

This shows \(T\) is an \(L^1\)-isometry on \(L^1(X) \cap L^p(X)\), since \(\{1_A: A \in \mathcal{F}\}\) is total and \(T\) maps disjointly supported indicators onto disjointly supported functions. Further, \(T\) extends by continuity to an \(L^1\)-isometry on all of \(L^1(X)\).

Now let \(A, B \in \mathcal{F}\) be disjoint such that \(\mu(A) < 1\) and \(\mu(B) < \infty\). Then \(\|1_A\| > \|1_A\|_1 = \|T_{1_A}\|_1\). Thus,
\[
\|1_A\|_p = \|1_A\|_1 = \|T_{1_A}\| = \|T_{1_A}\|_p.
\]
Put \(\epsilon > 0\) so small that
\[
\|T(1_A + \epsilon 1_B)\|_p > \|T(1_A + \epsilon 1_B)\|_1 \quad \text{and} \quad \|1_A + \epsilon 1_B\|_p > \|1_A + \epsilon 1_B\|_1.
\]
We have
\[
\|T_{1_A}\|_p + \epsilon^p \|T_{1_B}\|_p = \|T(1_A + \epsilon 1_B)\|_p = \|T(1_A + \epsilon 1_B)\|_p = \|1_A + \epsilon 1_B\|_p
\]
so \(\|T_{1_B}\|_p = \|1_B\|_p\). Therefore, \(T\) is also an \(L^p\)-isometry.

From this and the correct version of Lamperti's Theorem 3.1 [4] (in the case \(p = 1\)),

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where $\phi$ is an operator generated by a regular set isomorphism. We have

$$E(|h| \phi(\mathcal{F})) = \frac{d(\mu \circ \phi^{-1})}{d\mu} = E(|h|^p \phi(\mathcal{F}))$$

because $T$ is an $L^1$- and $L^p$-isometry. It should be pointed out that these equalities can also be written for $p = 2$ since $T$ maps functions with disjoint supports to functions with disjoint supports.

**Example.** $E(|H| \phi(\mathcal{F})) = E(|h|^p \phi(\mathcal{F})) \Rightarrow |h| = |h|^p$. The spaces $L^1(0, \infty) \cap L^p(0, \infty)$ and $L^1(-\infty, \infty) \cap L^p(-\infty, \infty)$ are isometrically isomorphic because there exists an invertible measure-preserving transformation $\tau: (-\infty, \infty) \to (0, \infty)$ (e.g., $\tau(x) = |x + [x]|$). Thus, consider now an operator $T: L^1(0, \infty) \cap L^p(0, \infty) \to L^1(-\infty, \infty) \cap L^p(-\infty, \infty)$ defined by

$$(Tf)(x) = \begin{cases} af(-x/\alpha) & \text{for } x < 0, \\ bf(x/\beta) & \text{for } x \geq 0, \end{cases}$$

where $\alpha, \beta, a, b$ are positive and satisfy $a\alpha + b\beta = a^p\alpha + b^p\beta = 1$. Obviously,

$$Tf = h\phi(f), \quad \text{where } h(x) = \begin{cases} a & \text{for } x < 0, \\ b & \text{for } x \geq 0, \end{cases}$$

$\phi(A) = (-\alpha A) \cup (\beta A) \quad (\gamma A = \{ \gamma x: x \in A \}, \gamma \in \mathbb{R})$. We have

$$E(|h|^p \phi(\mathcal{F})) = \frac{a^p\alpha + b^p\beta}{\alpha + \beta}.$$ 

Now let $1 < p_1 < p_2 < \infty$. If we choose $\alpha, \beta, a, b > 0$ such that $a^p_1\alpha + b^p_1\beta = a^p_2\alpha + b^p_2\beta = 1$, then $T$, defined as above, is an $L^p_1$- and $L^p_2$-isometry, but not necessarily an isometry for other $p \in [1, \infty)$. This example breaks down Corollary 3.1 in [4]. It should be noted that if $1 < p_1 < p_2 < \infty$ and $0 < a < 1 < b$ (or $a > 1 > b > 0$) then the solutions to our equations give us positive $\alpha, \beta$.

A function $\tau: X \to X$ is called an invertible measure-preserving transformation if $\tau^{-1}$ exists, $\tau$ and $\tau^{-1}$ are measurable, and $\mu(\tau(A)) = \mu(A)$ for all $A \in \mathcal{F}$.

**Corollary.** Let $T$ be a surjective isometry of $L^1(0, \infty) \cap L^p(0, \infty)$, $1 < p < \infty$. Then there exists an invertible measure-preserving transformation $\tau: (0, \infty) \to (0, \infty)$ and a measurable function $r$ with $|r| = 1$ such that

$$(Tf)(x) = r(x)\tau(x)).$$

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**References**


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