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A MEAN OSCILLATION INEQUALITY

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ABSTRACT. It is proved that $\|f^*\|_{\mathrm{BMO}} \leqslant \|f\|_{\mathrm{BMO}}$, where f^* is the decreasing rearrangement of a function $f \in \mathrm{BMO}([0,1])$. A generalization is given, as well as an example, showing the result fails for the symmetric decreasing rearrangement of a function on the circle.

In this paper we prove that

$$(*) \qquad \|f^*\|_{\mathrm{BMO}} \leqslant \|f\|_{\mathrm{BMO}},$$

where f^* is the decreasing rearrangement of the function $f\colon [0,1] \to \mathbf{R}$. The sharp constant one in this inequality refines a result contained in Theorem 3.1 of [**1**]. We also indicate generalizations, and finally mention symmetric decreasing rearrangement on the circle, for which $(*)$ fails.

The average of f over $E \subset [0,1]$, $|E| > 0$, will be denoted by $f_E = (1/|E|)\int_E f$. The definition of the norm in $(*)$ is

$$\|f\|_{\mathrm{BMO}} = \sup_{J \subset [0,1]} \frac{1}{|J|} \int_J |f - f_J|,$$

where J ranges over all subintervals of $[0,1]$. We need the following version of the rising sun lemma. See [**3**] or [**2**, p. 293] for the usual rising sun lemma.

LEMMA. *Let $f \in L^1([0,1])$ and suppose the average $f_{[0,1]} \leqslant \alpha$. Then there is a finite or countable set $\mathscr{L}$ of pairwise disjoint subintervals of $[0,1]$ such that $f_L = \alpha$ for each $L \in \mathscr{L}$, and $f \leqslant \alpha$ almost everywhere on $[0,1] \setminus \bigcup \mathscr{L}$.*

PROOF. It suffices to prove the case $\alpha = 0$. Define $F(x) = \int_0^x f$ on $[0,1]$. Choose $t_0 \geqslant 0$ as large as possible so that $F(t_0) = 0$. Let $\mathcal{O} = \{t \in [t_0, 1]\colon F(x) > F(t)$ for some $x > t\}$. Then $\mathcal{O}$ is open in $[t_0, 1]$, so $\mathcal{O} = \bigcup I_j$ for some disjoint intervals I_j open in $[t_0, 1]$. Defining $\mathscr{L}$ to be these intervals together with $[0, t_0)$ satisfies the lemma, as we now verify. On $[0, t_0)$ the average of f is 0 by definition. Let I_j have endpoints $t_0 \leqslant a_j < b_j \leqslant 1$.

If $F(b_j) > F(a_j)$ then $a_j \in \mathcal{O}$ by definition, so $a_j = t_0$. But, by hypothesis, $F(1) = f_{[0,1]} \leqslant 0 = F(t_0) = F(a_j) < F(b_j)$, so (by the intermediate value theorem) $F(c) = 0$ for some $b_j < c \leqslant 1$, contradicting the choice of t_0.

If $F(b_j) < F(a_j)$ choose $\tau \in [a_j, b_j]$ as large as possible so that

$$F(\tau) = \tfrac{1}{2}\big(F(a_j) + F(b_j)\big).$$

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Then $a_j < \tau < b_j$, so $\tau \in \mathcal{O}$, and, by definition, $F(x) > F(\tau)$ for some $x > \tau$. In fact, $x > b_j$ by the choice of τ, hence $b_j \in \mathcal{O}$ and $b_j < 1$. But then b_j is not an endpoint of I_j, contradiction.

We conclude $F(b_j) = F(a_j)$, whence the average $f_{I_j} = 0$, as desired. Finally, by the Lebesgue differentiation theorem, $f(t) = \lim_{x \downarrow t}(F(x) - F(t))/(x - t)$ for almost all $t \in [0, 1]$. Since $F(x) \leqslant F(t)$ when $x > t \in [t_0, 1] \setminus \mathcal{O}$, we conclude $f(t) \leqslant 0$ almost everywhere on $[0, 1] \setminus \bigcup \mathscr{L}$.

By visualizing the above proof the reader may notice that it had more to do with water levels than sun rays.

PROOF OF $(*)$. Fix an interval $J \subset [0, 1]$ and define $\alpha = f_J^*$. We assume $f_{[0,1]} \leqslant \alpha$ since an entirely symmetrical argument will cover the case $f_{[0,1]} \geqslant \alpha$. Apply the lemma to f and α and set $E = \bigcup \mathscr{L}$. To prove $(*)$ we have but to show

$$(**) \qquad \frac{1}{|J|}\int_J |f^* - \alpha| \leqslant \frac{1}{|E|}\int_E |f - \alpha|,$$

because then we may finish as follows:

$$\frac{1}{|E|}\int_E |f - \alpha| = \frac{\sum_{L \in \mathscr{L}} \int_L |f - \alpha|}{\sum_{L \in \mathscr{L}} |L|} \leqslant \sup_{L \in \mathscr{L}} \frac{1}{|L|}\int_L |f - \alpha| \leqslant \|f\|_{\mathrm{BMO}}.$$

Now $(**)$ can be proved in two steps.

First choose t as large as possible so that $(1/t)\int_0^t f^* = \alpha$. t exists because $f^*_{[0,1]} \leqslant \alpha$ and $f_J^* = \alpha$. We claim that

$$(1) \qquad \frac{1}{|J|}\int_J |f^* - \alpha| \leqslant \frac{1}{t}\int_0^t |f^* - \alpha|.$$

To see this, choose a point $\tau \in J$ such that $f^*(x) \geqslant \alpha$ for $x < \tau$ and $f^*(x) \leqslant \alpha$ for $x > \tau$. Let $J = [\tau - x_1, \tau + x_2]$ and $[0, t] = [\tau - y_1, \tau + y_2]$. Define

$$A \equiv \int_{\tau - x_1}^{\tau} |f^* - \alpha| = \int_{\tau}^{\tau + x_2} |f^* - \alpha|,$$

and

$$B \equiv \int_0^{\tau} |f^* - \alpha| = \int_{\tau}^{t} |f^* - \alpha|.$$

The monotonicity of $|f^* - \alpha|$ on $[0, \tau)$ implies $A/x_1 \leqslant B/y_1$. Similarly on $(\tau, t]$ we get $A/x_2 \leqslant B/y_2$. Taking reciprocals and adding (in the nontrivial cases) we get $(x_1 + x_2)/A \geqslant (y_1 + y_2)/B$ or $2A/(x_1 + x_2) \leqslant 2B/(y_1 + y_2)$, which is (1).

The second step is to prove

$$(2) \qquad \frac{1}{t}\int_0^t |f^* - \alpha| \leqslant \frac{1}{|E|}\int_E |f - \alpha|.$$

In fact, we will show $t \geqslant |E|$ and $\int_0^t |f^* - \alpha| = \int_E |f - \alpha|$.

By properties of f^*, the identity $\int_0^{|E|} f^* \geqslant \int_E f$ holds for any set E. For our E it implies

$$\frac{1}{|E|}\int_0^{|E|} f^* \geqslant \frac{1}{|E|}\int_E f = \alpha,$$

whence $t \geqslant |E|$ by the definition of t and monotonicity of f^*.

Finally, recall that $f \leqslant \alpha$ almost everywhere on $[0,1] \setminus E$. Hence, we can argue

$$\int_0^t |f^* - \alpha| = 2\int_{\{x: f^*(x) > \alpha\}} f^* - \alpha = 2\int_{\{x: f(x) > \alpha\}} f - \alpha$$
$$= 2\int_{\{x: f(x) > \alpha\} \cap E} f - \alpha = \int_E |f - \alpha|.$$

This concludes the proof of $(*)$.

The existence of the intermediate quantity $(1/t)\int_0^t |f^* - \alpha|$ (the "worst case" for f^*) and the *apparently* distinct reasons for the inequalities on either side of it struck the author as rather arresting. Furthermore, there appeared to be lurking a curious symmetry between the behaviour of the numerators and denominators in the whole proof. Both of these points were happily resolved by

THEOREM. *Let $f \in L^1([0,1])$. Suppose $J \subset [0,1]$ is an interval and $E \subset [0,1]$ is a set such that*

(i) $f_J^* = f_E \equiv \alpha$ *and*

(ii) $\int_J |f^* - \alpha| \leqslant \int_E |f - \alpha|$.

Suppose further that F, G: $[0,\infty) \to [0,\infty)$ are such that $F(\lambda)/\lambda$ is increasing and $G(\lambda)/\lambda$ is decreasing for $\lambda > 0$, $F(0) = 0$, and $F \circ |f - \alpha|$, $G \circ |f - \alpha| \in L^1([0,1])$. Then

$$\frac{\int_J F \circ |f^* - \alpha|}{\int_J G \circ |f^* - \alpha|} \leqslant \frac{\int_E F \circ |f - \alpha|}{\int_E G \circ |f - \alpha|} \tag{3}$$

(*where we take* $0/0 = 0$ *and* $x/0 = \infty$, $x > 0$).

The author has a proof which involves writing the integrals in terms of distribution functions of the kind $m(\lambda) = \int_{\{t: f(t) - \alpha > \lambda\} \cap E} f - \alpha$ and using integration by parts. We can choose $F(\lambda) = \lambda^p$, $1 \leqslant p < \infty$, $G(\lambda) = 1$ and, using the same set E as in our earlier argument, obtain $(*)$ with exponent p in place of 1.

We now consider functions of bounded mean oscillation on the circle $T \equiv \mathbf{R}/2\pi\mathbf{Z}$. For $f \in \mathrm{BMO}(T)$ define

$$\|f\| = \sup_J \frac{1}{|J|} \int_J |f - f_J|,$$

where J ranges over all intervals in T. Let $f^\#$ denote the symmetric decreasing rearrangement of f. Making use of $(*)$, it can be shown that $\|f^\#\| \leqslant 2\|f\|$. However, there are functions f such that $\|f^\#\| > \|f\|$. We give an example, omitting the computations.

Define $f(0) = f(1) = f(-1) = 0$, $f(\frac{1}{3}) = f(-\frac{1}{3}) = 1$ and interpolate linearly for the remaining $\theta \in [-\pi, \pi]$. Then $f^\#$ is the piecewise linear function with corners $(-1, 0)$, $(0, 1)$, $(1, 0)$, and it can be shown that $\|f^\#\| > \|f\|$.

The failure of the inequality $\|f^\#\| \leqslant \|f\|$ is due to the fact that the supremum $\|f^\#\|$ may be achieved for an interval J on which $f^\#$ is not monotone. This makes it possible to construct an (equimeasurable) "perturbation" f for which $\|f\| < \|f^\#\|$ (as in the above example).

We conclude with two problems:

(1) For $f \in \mathrm{BMO}(T)$ are there any equimeasurable rearrangements g for which $\|g\|$ is minimal? If so, describe them.

(2) What is the best constant c such that $\|f^{\#}\| \leqslant c\|f\|$ for all $f \in \mathrm{BMO}(T)$?

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