A MEAN OSCILLATION INEQUALITY

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Abstract. It is proved that \( \|f^*\|_{BMO} \leq \|f\|_{BMO} \), where \( f^* \) is the decreasing rearrangement of a function \( f \in BMO([0,1]) \). A generalization is given, as well as an example, showing the result fails for the symmetric decreasing rearrangement of a function on the circle.

In this paper we prove that
\[
(\ast) \quad \|f^*\|_{BMO} \leq \|f\|_{BMO},
\]
where \( f^* \) is the decreasing rearrangement of the function \( f: [0,1] \to \mathbb{R} \). The sharp constant one in this inequality refines a result contained in Theorem 3.1 of [1]. We also indicate generalizations, and finally mention symmetric decreasing rearrangement on the circle, for which (\ast) fails.

The average of \( f \) over \( E \subset [0,1], |E| > 0 \), will be denoted by \( f_E = (1/|E|) \int_E f \). The definition of the norm in (\ast) is
\[
\|f\|_{BMO} = \sup_{J \subset [0,1]} \frac{1}{|J|} \int_J |f - f_J|,
\]
where \( J \) ranges over all subintervals of \([0,1]\). We need the following version of the rising sun lemma. See [3] or [2, p. 293] for the usual rising sun lemma.

Lemma. Let \( f \in L^1([0,1]) \) and suppose the average \( f_{[0,1]} \leq \alpha \). Then there is a finite or countable set \( \mathcal{L} \) of pairwise disjoint subintervals of \([0,1]\) such that \( f_L = \alpha \) for each \( L \in \mathcal{L} \), and \( f \leq \alpha \) almost everywhere on \([0,1] \setminus \bigcup \mathcal{L} \).

Proof. It suffices to prove the case \( \alpha = 0 \). Define \( F(x) = \int_0^x f \) on \([0,1] \). Choose \( t_0 > 0 \) as large as possible so that \( F(t_0) = 0 \). Let \( \mathcal{O} = \{ t \in [t_0,1]: F(x) > F(t) \text{ for some } x > t \} \). Then \( \mathcal{O} \) is open in \([t_0,1] \), so \( \mathcal{O} = \bigcup I_j \) for some disjoint intervals \( I_j \) open in \([t_0,1] \). Defining \( \mathcal{L} \) to be these intervals together with \([0,t_0] \) satisfies the lemma, as we now verify. On \([0,t_0] \) the average of \( f \) is 0 by definition. Let \( I_j \) have endpoints \( t_0 \leq a_j < b_j \leq 1 \).

If \( F(b_j) > F(a_j) \) then \( a_j \in \mathcal{O} \) by definition, so \( a_j = t_0 \). But, by hypothesis, \( F(1) = \int_{[0,1]} f \leq 0 = F(t_0) = F(a_j) < F(b_j) \), so (by the intermediate value theorem) \( F(c) = 0 \) for some \( b_j < c \leq 1 \), contradicting the choice of \( t_0 \).

If \( F(b_j) < F(a_j) \) choose \( \tau \in [a_j, b_j] \) as large as possible so that
\[
F(\tau) = \frac{1}{2} \left( F(a_j) + F(b_j) \right).
\]
Then \( a_j < \tau < b_j \), so \( \tau \in \emptyset \), and, by definition, \( F(x) > F(\tau) \) for some \( x > \tau \). In fact, \( x > b_j \) by the choice of \( \tau \), hence \( b_j \in \emptyset \) and \( b_j < 1 \). But then \( b_j \) is not an endpoint of \( I_j \), contradiction.

We conclude \( F(b_j) = F(a_j) \), whence the average \( f_j = 0 \), as desired. Finally, by the Lebesgue differentiation theorem, \( f(t) = \lim_{x \to t} \frac{F(x) - F(t)}{x - t} \) for almost all \( t \in [0,1] \). Since \( F(x) \leq F(t) \) when \( x > t \in [t_0,1] \setminus \emptyset \), we conclude \( f(t) \leq 0 \) almost everywhere on \([0,1] \setminus \cup \mathcal{L} \).

By visualizing the above proof the reader may notice that it had more to do with water levels than sun rays.

**Proof of (\star \star)***. Fix an interval \( J \subset [0,1] \) and define \( \alpha = f_J^* \). We assume \( f_{[0,1]} \leq \alpha \) since an entirely symmetrical argument will cover the case \( f_{[0,1]} \geq \alpha \). Apply the lemma to \( f \) and \( \alpha \) and set \( E = \cup \mathcal{L} \). To prove (\star) we have but to show

\[
\frac{1}{|J|} \int_J |f^* - \alpha| \leq \frac{1}{|E|} \int_E |f - \alpha|,
\]

because then we may finish as follows:

\[
\frac{1}{|E|} \int_E |f - \alpha| = \frac{\sum_{L \in \mathcal{L}} |f - \alpha|}{\sum_{L \in \mathcal{L}} |L|} \leq \sup_{L \in \mathcal{L}} \frac{1}{|L|} \int_L |f - \alpha| \leq \|f\|_{BMO}.
\]

Now (\star \star) can be proved in two steps.

First choose \( t \) as large as possible so that \( (1/t)\int_0^t f^* = \alpha \). \( t \) exists because \( f_{[0,1]} \leq \alpha \) and \( f_J^* = \alpha \). We claim that

\[
\frac{1}{|J|} \int_J |f^* - \alpha| \leq \frac{1}{t} \int_0^t |f^* - \alpha|.
\]

To see this, choose a point \( \tau \in J \) such that \( f^*(x) \geq \alpha \) for \( x < \tau \) and \( f^*(x) \leq \alpha \) for \( x > \tau \). Let \( J = [\tau - x_1, \tau + x_2] \) and \( [0, t] = [\tau - y_1, \tau + y_2] \). Define

\[ A = \int_{\tau - x_1}^\tau |f^* - \alpha| = \int_{\tau}^{\tau + x_2} |f^* - \alpha|, \]

and

\[ B = \int_0^t |f^* - \alpha| = \int_\tau^t |f^* - \alpha|. \]

The monotonicity of \( |f^* - \alpha| \) on \([0, \tau]\) implies \( A/x_1 \leq B/y_1 \). Similarly on \((\tau, t]\) we get \( A/x_2 \leq B/y_2 \). Taking reciprocals and adding (in the nontrivial cases) we get

\[
(x_1 + x_2)/A \geq (y_1 + y_2)/B \quad \text{or} \quad 2A/(x_1 + x_2) \leq 2B/(y_1 + y_2), \]

which is (1).

The second step is to prove

\[
\frac{1}{t} \int_0^t |f^* - \alpha| \leq \frac{1}{|E|} \int_E |f - \alpha|.
\]

In fact, we will show \( t \geq |E| \) and \( \int_E |f^* - \alpha| = \int_E |f - \alpha| \).

By properties of \( f^* \), the identity \( \int_0^t f^* \geq \int_E f \) holds for any set \( E \). For our \( E \) it implies

\[
\frac{1}{|E|} \int_0^t f^* \geq \frac{1}{|E|} \int_E f = \alpha,
\]

whence \( t \geq |E| \) by the definition of \( t \) and monotonicity of \( f^* \).
Finally, recall that $f \leq \alpha$ almost everywhere on $[0, 1] \setminus E$. Hence, we can argue

$$
\int_0^t |f^* - \alpha| = 2 \int_{\{x : f^*(x) > \alpha\}} f^* - \alpha = 2 \int_{\{x : f(x) > \alpha\}} f - \alpha
$$

$$
= 2 \int_{\{x : f(x) > \alpha\} \cap E} f - \alpha = \int_E |f - \alpha|.
$$

This concludes the proof of $(\ast)$.

The existence of the intermediate quantity $(1/t)\int_0^t |f^* - \alpha|$ (the “worst case” for $f^*$) and the apparently distinct reasons for the inequalities on either side of it struck the author as rather arresting. Furthermore, there appeared to be lurking a curious symmetry between the behaviour of the numerators and denominators in the whole proof. Both of these points were happily resolved by

**Theorem.** Let $f \in L^1([0, 1])$. Suppose $J \subset [0, 1]$ is an interval and $E \subset [0, 1]$ is a set such that

(i) $f^*_E = f \equiv \alpha$ and

(ii) $\int_J |f^* - \alpha| \leq \int_E |f - \alpha|.$

Suppose further that $F, G : [0, \infty) \to [0, \infty)$ are such that $F(\lambda)/\lambda$ is increasing and $G(\lambda)/\lambda$ is decreasing for $\lambda > 0$, $F(0) = 0$, and $F \circ |f - \alpha|, G \circ |f - \alpha| \in L^1([0, 1])$. Then

$$
\int_J F \circ |f^* - \alpha| \leq \int_E F \circ |f - \alpha|,
$$

$$
\int_J G \circ |f^* - \alpha| \leq \int_E G \circ |f - \alpha|
$$

(where we take $0/0 = 0$ and $x/0 = \infty$, $x > 0$).

The author has a proof which involves writing the integrals in terms of distribution functions of the kind $m(\lambda) = \int_{\{x : f(x) > \lambda\} \cap E} f - \alpha$ and using integration by parts. We can choose $F(\lambda) = \lambda^p$, $1 < p < \infty$, $G(\lambda) = 1$ and, using the same set $E$ as in our earlier argument, obtain $(\ast)$ with exponent $p$ in place of 1.

We now consider functions of bounded mean oscillation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let $f \in \text{BMO}(\mathbb{T})$ define

$$
\|f\| = \sup_J \frac{1}{|J|} \int_J |f - f_J|,
$$

where $J$ ranges over all intervals in $\mathbb{T}$. Let $f^\#$ denote the symmetric decreasing rearrangement of $f$. Making use of $(\ast)$, it can be shown that $\|f^\#\| \leq 2\|f\|$. However, there are functions $f$ such that $\|f^\#\| > \|f\|$. We give an example, omitting the computations.

Define $f(0) = f(1) = f(-1) = 0, f(\frac{1}{2}) = f(- \frac{1}{2}) = 1$ and interpolate linearly for the remaining $\theta \in [-\pi, \pi]$. Then $f^\#$ is the piecewise linear function with corners $(-1, 0), (0, 1), (1, 0)$, and it can be shown that $\|f^\#\| > \|f\|$.

The failure of the inequality $\|f^\#\| \leq \|f\|$ is due to the fact that the supremum $\|f^\#\|$ may be achieved for an interval $J$ on which $f^\#$ is not monotone. This makes it possible to construct an (equimeasurable) “perturbation” $f$ for which $\|f\| < \|f^\#\|$ (as in the above example).
We conclude with two problems:

(1) For $f \in \text{BMO}(T)$ are there any equimeasurable rearrangements $g$ for which $\|g\|$ is minimal? If so, describe them.

(2) What is the best constant $c$ such that $\|f^\#\| \leq c\|f\|$ for all $f \in \text{BMO}(T)$?

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References