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EVERY UNIFORMLY CONTINUOUS CENTERED SEMIGROUP IS NORMAL

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ABSTRACT. A well-known result on commutators of operators is used to prove the assertion in the title.

Consider a strongly continuous semigroup (S_t: $t \geqslant 0$) of continuous linear operators on a Hilbert space $\mathscr{H}$. Let U_tP_t be the canonical polar factorization of S_t. The semigroup (S_t) is said to be *centered* if the partially isometric factor (U_t) of (S_t) is a semigroup. Examples of strongly continuous nonnormal centered semigroups are given in [**1**]. The purpose of this note is to demonstrate the truth of the assertion given in the title.

THEOREM. *Every uniformly continuous centered semigroup is normal.*

PROOF. Let (S_t) be a uniformly continuous centered semigroup with (bounded) generator A; that is, $S_t = e^{tA}$. We need only show that the operator A is normal and we shall do so. Consider the power series expansions of $S_t^*S_t$ and $S_tS_t^*$:

$$S_t^*S_t = I + t(A + A^*) + t^2(A^2 + 2A^*A + A^{*2}) + O(t^3)$$

and

$$S_tS_t^* = I + t(A + A^*) + t^2(A^2 + 2AA^* + A^{*2}) + O(t^3).$$

Since (S_t) is centered, $S_t^*S_t$ commutes with $S_rS_r^*$ for all nonnegative t and r [**1**, Lemma 2]. Consequently, the coefficients in the power series expansions of $S_t^*S_t$ and $S_tS_t^*$ form an abelian set of operators. In particular, $A + A^*$ commutes with $A^2 + 2A^*A + A^{*2}$ and $A^2 + 2AA^* + A^{*2}$, and with their difference $2(A^*A - AA^*)$. That A is normal follows from the next lemma.

LEMMA. *An operator A on a Hilbert space $\mathscr{H}$ is normal if $A + A^*$ commutes with $A^*A - AA^*$.*

PROOF. Write $A = H + iK$ where $H = (A + A^*)/2$ and $K = (A - A^*)/2i$. If $A + A^*$ commutes with $A^*A - AA^*$, then H commutes with $HK - KH$. Kleinecke's result [**2**] on commutators implies that 0 is the only element in the spectrum $HK - KH$. Since H and K are Hermitian, this in turn implies that $HK - KH = 0$ or equivalently that A is normal.

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Using the Theorem one can easily construct examples of semigroups for which the generator has nice algebraic properties not inherited by the semigroup itself. For example, if A is a nonnormal centered operator [3] (that is, $\{A^{*n}A^n, A^mA^{*m}: m, n$ nonnegative integers$\}$ is abelian), then (e^{tA}) is nonnormal and, consequently, not centered. Similarly, if A is a nonnormal, subnormal operator, then (e^{tA}) is not centered.

We showed in the proof of the Theorem that A is normal whenever the coefficients of the power series expansion of $S_t^*S_t$ commute with those of $S_tS_t^*$. These coefficients also arise in [4] where Lambert shows that A is subnormal if each of the coefficients of $S_t^*S_t$ is nonnegative definite.

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