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CYCLIC NEVANLINNA CLASS FUNCTIONS IN BERGMAN SPACES

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ABSTRACT. Let f be a function which is in both the Bergman space A^p ($p \geq 1$) and the Nevanlinna class N. We show that if f is expressed as the quotient of H^∞ functions, then the inner part of its denominator is cyclic. As a corollary, we obtain that f is cyclic if and only if the inner part of its numerator is cyclic. These results extend those of Berman, Brown, and Cohn [**2**]. Using more difficult methods, they have obtained them for the case $f \in A^2 \cap N$. Finally, we show that the condition $|f(z)| \geq \delta(1-|z|)^c$ ($z \in D; \delta, c$ positive constants) is sufficient for cyclicity for $f \in A^p \cap N$, which answers a question of Aharonov, Shapiro, and Shields [**1**].

1. Introduction. For $1 \leq p < \infty$, let A^p denote the Bergman space consisting of those functions f holomorphic on the open unit disk D satisfying $\|f\|_p < \infty$, where

$$\|f\|_p^p = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^p r\,dr\,d\theta.$$

The norm $\| \;\|_p$ makes A^p into a Banach space in which the polynomials are dense.

For $f \in A^p$ define $[f]$ by $[f] = A^p$-closure of $\{pf : p \text{ is a polynomial}\}$. We say that a function $f \in A^p$ is cyclic in A^p provided it is cyclic for the forward shift operator T_z on A^p, where T_z is defined by $(T_z f)(z) = zf(z)$. Thus f is cyclic in A^p if $[f] = A^p$. It is not difficult to verify that $[f] = A^p$-closure of $\{hf : h \in H^\infty(D)\}$ and that the following are equivalent.

(a) f is cyclic in A^p.

(b) There is a sequence $\{p_n\}$ of polynomials such that $\|p_n f - 1\|_p \to 0$.

(c) There is a sequence $\{h_n\}$ of H^∞ functions such that $\|h_n f - 1\|_p \to 0$.

Let $S\mu$ denote the singular inner function induced by μ; that is,

$$S\mu(z) = \exp\left(-\int_T \frac{\omega + z}{\omega - z}\,d\mu(\omega)\right).$$

Here, of course, μ is a positive finite Borel measure on the unit circle T singular with respect to Lebesgue measure. The Nevanlinna class N consists of those functions f holomorphic on D which satisfy

$$\sup_{r<1}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta < \infty.$$

If $f \in N$, then f factors as $B\gamma S\nu/\phi S\mu$, where B is a Blaschke product, γ and ϕ are outer functions in H^∞, and where ν and μ are mutually singular measures ($\nu \perp \mu$) [**3**, p. 25].

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We now establish two easy propositions for future reference.

PROPOSITION 1. *If $\gamma \in H^\infty$ is outer, then γ is cyclic in A^p.*

PROOF. That γ is cyclic in H^p follows from Beurling's Theorem [**3**, p. 114]; however, convergence in H^p implies convergence in A^p.

PROPOSITION 2. *$f \in H^\infty$ is cyclic in A^p if and only if its inner part is cyclic in A^p.*

PROOF. It is clear that cyclic vectors in A^p must be nonzero; hence, f factors as $\gamma S\mu$ where γ is outer and in H^∞. We show that $[f] = [S\mu]$. Since f is an H^∞ multiple of its inner part $S\mu$, $[S\mu] \supset [f]$. The outer function γ is cyclic, so there is a sequence $\{p_n\}$ of polynomials such that $\|p_n\gamma - 1\|_p \to 0$. Now, making use of the fact that $\|hg\|_p \le \|h\|_\infty \|g\|_p$ for $g \in A^p$ and $h \in H^\infty$, we find $\|p_n\gamma S\mu - S\mu\|_p \le \|p_n\gamma - 1\|_p$ so that $S\mu \in [f]$. It follows that $[S\mu] = [f]$.

We have the following characterization of cyclic inner functions in A^p.

THEOREM. *$S\mu$ is cyclic in A^p if and only if μ places no mass on any Carleson set.*

Necessity was proved by H. S. Shapiro [**8**], sufficiency by B. Korenblum [**4, 6**] and (independently) J. Roberts [**7**, **9**]. Carleson sets are certain compact sets of measure zero in T. Specifically, a compact set $K \subset T$ is Carleson if $\int_T \log \rho_K(\omega)\, dm(\omega) > -\infty$, where $\rho_K(\omega) = \operatorname{dist}(\omega, K)$ and m is Lebesgue measure on T.

2. Results. We postpone the proof of the following theorem until after presenting some of its consequences.

THEOREM 1. *If $f \in A^p \cap N$, then $f = B\gamma S\nu/\phi S\mu$ where $S\mu$ is cyclic in A^p.*

COROLLARY 1. *$f \in A^p \cap N$ is cyclic in A^p if and only if $f = \gamma S\nu/\phi S\mu$ where $S\nu$ is cyclic.*

PROOF. If $S\nu$ is cyclic then $\gamma S\nu$ is cyclic (Proposition 2). But $\gamma S\nu = (\phi S\mu)f$ so that $[f]$ contains the cyclic vector $\gamma S\nu$. Thus f is cyclic.

Conversely, suppose f is cyclic. Since cyclic vectors are nonzero, f factors as $\gamma S\nu/\phi S\mu$. Let $\{p_n\}$ be a sequence of polynomials such that $\|p_n f - 1\|_p \to 0$. Then

$$\begin{aligned} \|p_n\gamma S\nu - \phi S\mu\|_p &= \left\| \phi S\mu \left(p_n \frac{\gamma S\nu}{\phi S\mu} - 1 \right) \right\|_p \\ &\le \|\phi\|_\infty \|p_n f - 1\|_p; \end{aligned}$$

consequently, $\phi S\mu \in [\gamma S\nu]$. However, $\phi S\mu$ is cyclic by Theorem 1; thus $S\nu$ is cyclic.

COROLLARY 2. *If $f \in A^p \cap N$ and if $1/f \in A^p$, then f is cyclic in A^p.*

PROOF. Let $f = \gamma S\nu/\phi S\mu$. $S\nu$ is cyclic since $1/f \in A^p$; therefore, f is cyclic by Corollary 1.

REMARK. Using different methods, Berman, Brown, and Cohn [**2**] have obtained Theorem 1 as well as Corollaries 1 and 2 for the case $f \in A^2 \cap N$.

THEOREM 2. *If $f \in A^p \cap N$ satisfies $|f(z)| \geq \delta(1-|z|)^c$ for $z \in D$ (δ, c positive constants) then f is cyclic in A^p.*

PROOF. Again, let $f = \gamma S\nu/\phi S\mu$. Note that f has an analytic nth root since it is nonzero on D. Choose n large enough so that $c/n \leq 1/2p$. It is easy to check that both $f^{1/n}$ and $f^{-1/n}$ are in A^p. We see that $f^{1/n}$ is cyclic by Corollary 2, and consequently ν/n places no mass on any Carleson set (Corollary 1). Therefore, ν places no mass on any Carleson set and $S\nu$ is cyclic. Now, f is cyclic by Corollary 1.

3. Proof of Theorem 1. An inner function q is said to be *B-inner* if it divides every inner function in $[q]$. The following theorem was established independently by James Roberts [**7**] and Pat Ahern [**9**].

THEOREM 3. *$S\mu = S\mu_b S\mu_c$ where $S\mu_b$ is B-inner and $S\mu_c$ is cyclic.*

Ahern has shown that the B-inner, cyclic factorization in the Bergman spaces is a corollary of the Shapiro-Korenblum-Roberts characterization of cyclic inner functions, while Roberts has proved the existence of such a factorization in a somewhat more general setting without using the cyclic inner function characterization. Both Roberts and Ahern have shown that in the Bergman spaces, $S\mu$ is B-inner if and only if μ is concentrated on a countable union of Carleson sets.

PROOF OF THEOREM 1. Recall $f = B\gamma S\nu/\phi S\mu$ where $\nu \perp \mu$. Let $\{p_n\}$ be a sequence of polynomials such that $\|p_n - f\|_p \to 0$. Then

$$\begin{aligned}\|p_n\phi S\mu - B\gamma S\nu\|_p &= \|\phi S\mu(p_n - B\gamma S\nu/\phi S\mu)\|_p \\ &\leq \|\phi\|_\infty \|p_n - f\|_p;\end{aligned}$$

consequently, $B\gamma S\nu \in [\phi S\mu]$. But γ is outer and in H^∞; therefore, we have, just as in the proof of Proposition 2, $BS\nu \in [B\gamma S\nu]$. Hence, $BS\nu \in [\phi S\mu]$. Thus there is a sequence $\{q_n\}$ of polynomials such that $\|q_n\phi S\mu - BS\nu\|_p \to 0$.

We have $\|(q_n\phi S\mu_c)S\mu_b - BS\nu\|_p \to 0$ so that in fact, $BS\nu \in [S\mu_b]$. Now, by the definition of B-inner, there exists an inner function q such that $BS\nu = qS\mu_b$. It follows that $S\mu_b | S\nu$; however, $\mu_b \perp \nu$ and hence $S\mu_b = 1$. Thus $S\mu = S\mu_c$ is cyclic.

REMARKS. 1. Corollary 1 may be proved directly using an argument similar to the one used in the proof of Theorem 1.

2. Since Shapiro-Korenblum-Roberts characterization of cyclic inner functions remains valid in the Bergman spaces A^p_α with $\alpha > -1$ and $0 < p < \infty$, our results extend readily to these spaces. Here the subscript α indicates that the Bergman norm is induced by the weighted measure $(1-r)^\alpha r\, dr\, d\theta$.

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