ABSTRACT. Let \( f \) be a function which is in both the Bergman space \( A^p \) \((p \geq 1)\) and the Nevanlinna class \( N \). We show that if \( f \) is expressed as the quotient of \( H^\infty \) functions, then the inner part of its denominator is cyclic. As a corollary, we obtain that \( f \) is cyclic if and only if the inner part of its numerator is cyclic. These results extend those of Berman, Brown, and Cohn [2]. Using more difficult methods, they have obtained them for the case \( f \in A^2 \cap N \). Finally, we show that the condition \( |f(z)| \geq \delta (1 - |z|^c) \) \((z \in D; \delta, c \text{ positive constants})\) is sufficient for cyclicity for \( f \in A^p \cap N \), which answers a question of Aharonov, Shapiro, and Shields [1].

1. Introduction. For \( 1 \leq p < \infty \), let \( A^p \) denote the Bergman space consisting of those functions \( f \) holomorphic on the open unit disk \( D \) satisfying \( \|f\|_p < \infty \), where

\[
\|f\|_p^p = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^p r \, dr \, d\theta.
\]

The norm \( \| \cdot \|_p \) makes \( A^p \) into a Banach space in which the polynomials are dense.

For \( f \in A^p \) define \([f]\) by \([f]\) = \( A^p\)-closure of \( \{pf: p \text{ is a polynomial}\} \). We say that a function \( f \in A^p \) is cyclic in \( A^p \) provided it is cyclic for the forward shift operator \( T_z \) on \( A^p \), where \( T_z \) is defined by \((T_z f)(z) = zf(z)\). Thus \( f \) is cyclic in \( A^p \) if \([f]\) = \( A^p \). It is not difficult to verify that \([f]\) = \( A^p\)-closure of \( \{hf: h \in H^\infty(D)\} \) and that the following are equivalent.

(a) \( f \) is cyclic in \( A^p \).
(b) There is a sequence \( \{p_n\} \) of polynomials such that \( \|p_n f - 1\|_p \to 0 \).
(c) There is a sequence \( \{h_n\} \) of \( H^\infty \) functions such that \( \|h_n f - 1\|_p \to 0 \).

Let \( S\mu \) denote the singular inner function induced by \( \mu \); that is,

\[
S\mu(z) = \exp \left( - \int_T \frac{\omega + z}{\omega - z} d\mu(\omega) \right).
\]

Here, of course, \( \mu \) is a positive finite Borel measure on the unit circle \( T \) singular with respect to Lebesgue measure. The Nevanlinna class \( N \) consists of those functions \( f \) holomorphic on \( D \) which satisfy

\[
\sup_{r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty.
\]

If \( f \in N \), then \( f \) factors as \( B^\gamma S\nu/\phi S\mu \), where \( B \) is a Blaschke product, \( \gamma \) and \( \phi \) are outer functions in \( H^\infty \), and where \( \nu \) and \( \mu \) are mutually singular measures \((\nu \perp \mu)\) [3, p. 25].
We now establish two easy propositions for future reference.

**Proposition 1.** If $\gamma \in H^\infty$ is outer, then $\gamma$ is cyclic in $A^p$.

**Proof.** That $\gamma$ is cyclic in $H^p$ follows from Beurling's Theorem [3, p. 114]; however, convergence in $H^p$ implies convergence in $A^p$.

**Proposition 2.** $f \in H^\infty$ is cyclic in $A^p$ if and only if its inner part is cyclic in $A^p$.

**Proof.** It is clear that cyclic vectors in $A^p$ must be nonzero; hence, $f$ factors as $\gamma S\mu$ where $\gamma$ is outer and in $H^\infty$. We show that $[f] = [S\mu]$. Since $f$ is an $H^\infty$ multiple of its inner part $S\mu$, $[S\mu] \supset [f]$. The outer function $\gamma$ is cyclic, so there is a sequence $\{p_n\}$ of polynomials such that $\|p_n\gamma - 1\|_p \to 0$. Now, making use of the fact that $\|hg\|_p \leq \|h\|_\infty \|g\|_p$ for $g \in A^p$ and $h \in H^\infty$, we find $\|p_n \gamma S\mu - S\mu\|_p \leq \|p_n \gamma - 1\|_p$ so that $S\mu \in [f]$. It follows that $[S\mu] = [f]$.

We have the following characterization of cyclic inner functions in $A^p$.

**Theorem.** $S\mu$ is cyclic in $A^p$ if and only if $\mu$ places no mass on any Carleson set.

Necessity was proved by H. S. Shapiro [8], sufficiency by B. Korenblum [4, 6] and (independently) J. Roberts [7, 9]. Carleson sets are certain compact sets of measure zero in $T$. Specifically, a compact set $K \subset T$ is Carleson if $\int_T \log \rho_K(\omega) dm(\omega) > -\infty$, where $\rho_K(\omega) = \text{dist}(\omega, K)$ and $m$ is Lebesgue measure on $T$.

2. Results. We postpone the proof of the following theorem until after presenting some of its consequences.

**Theorem 1.** If $f \in A^p \cap N$, then $f = B\gamma S\nu/\phi S\mu$ where $S\mu$ is cyclic in $A^p$.

**Corollary 1.** $f \in A^p \cap N$ is cyclic in $A^p$ if and only if $f = \gamma S\nu/\phi S\mu$ where $S\nu$ is cyclic.

**Proof.** If $S\nu$ is cyclic then $\gamma S\nu$ is cyclic (Proposition 2). But $\gamma S\nu = (\phi S\mu)f$ so that $[f]$ contains the cyclic vector $\gamma S\nu$. Thus $f$ is cyclic.

Conversely, suppose $f$ is cyclic. Since cyclic vectors are nonzero, $f$ factors as $\gamma S\nu/\phi S\mu$. Let $\{p_n\}$ be a sequence of polynomials such that $\|p_n f - 1\|_p \to 0$. Then

$$\|p_n \gamma S\nu - \phi S\mu\|_p = \left\|\phi S\mu \left(\frac{p_n \gamma S\nu}{\phi S\mu} - 1\right)\right\|_p \leq \|\phi\|_\infty \|p_n f - 1\|_p;$$

consequently, $\phi S\mu \in [\gamma S\nu]$. However, $\phi S\mu$ is cyclic by Theorem 1; thus $S\nu$ is cyclic.

**Corollary 2.** If $f \in A^p \cap N$ and if $1/f \in A^p$, then $f$ is cyclic in $A^p$.

**Proof.** Let $f = \gamma S\nu/\phi S\mu$. $S\nu$ is cyclic since $1/f \in A^p$; therefore, $f$ is cyclic by Corollary 1.

**Remark.** Using different methods, Berman, Brown, and Cohn [2] have obtained Theorem 1 as well as Corollaries 1 and 2 for the case $f \in A^2 \cap N$. 
THEOREM 2. If \( f \in A^p \cap N \) satisfies \( |f(z)| \geq \delta (1 - |z|)^c \) for \( z \in D \) (\( \delta, c \) positive constants) then \( f \) is cyclic in \( A^p \).

PROOF. Again, let \( f = \gamma S \nu / \phi S \mu \). Note that \( f \) has an analytic \( n \)th root since it is nonzero on \( D \). Choose \( n \) large enough so that \( c/n < 1/2p \). It is easy to check that both \( f^{1/n} \) and \( f^{-1/n} \) are in \( A^p \). We see that \( f^{1/n} \) is cyclic by Corollary 2, and consequently \( \nu/n \) places no mass on any Carleson set (Corollary 1). Therefore, \( \nu \) places no mass on any Carleson set and \( S \nu \) is cyclic. Now, \( f \) is cyclic by Corollary 1.

3. Proof of Theorem 1. An inner function \( q \) is said to be \( B \)-inner if it divides every inner function in \([q]\). The following theorem was established independently by James Roberts [7] and Pat Ahern [9].

**THEOREM 3.** \( S \mu = S \mu_b S \mu_c \) where \( S \mu_b \) is \( B \)-inner and \( S \mu_c \) is cyclic.

Ahern has shown that the \( B \)-inner, cyclic factorization in the Bergman spaces is a corollary of the Shapiro-Korenblum-Roberts characterization of cyclic inner functions, while Roberts has proved the existence of such a factorization in a somewhat more general setting without using the cyclic inner function characterization. Both Roberts and Ahern have shown that in the Bergman spaces, \( S \mu \) is \( B \)-inner if and only if \( \mu \) is concentrated on a countable union of Carleson sets.

**PROOF OF THEOREM 1.** Recall \( f = B \gamma S \nu / \phi S \mu \) where \( \nu \perp \mu \). Let \( \{p_n\} \) be a sequence of polynomials such that \( \|p_n - f\|_p \to 0 \). Then

\[
\|p_n \phi S \mu - B \gamma S \nu\|_p = \|\phi S \mu (p_n - B \gamma S \nu / \phi S \mu)\|_p \\
\leq \|\phi\|_\infty \|p_n - f\|_p;
\]

consequently, \( B \gamma S \nu \in [\phi S \mu] \). But \( \gamma \) is outer and in \( H^\infty \); therefore, we have, just as in the proof of Proposition 2, \( BS \nu \in [B \gamma S \nu] \). Hence, \( BS \nu \in [\phi S \mu] \). Thus there is a sequence \( \{q_n\} \) of polynomials such that \( \|q_n \phi S \mu - BS \nu\|_p \to 0 \).

We have \( \|(q_n \phi S \mu_c)S \mu_b - BS \nu\|_p \to 0 \) so that in fact, \( BS \nu \in [S \mu_b] \). Now, by the definition of \( B \)-inner, there exists an inner function \( q \) such that \( BS \nu = q S \mu_b \). It follows that \( S \mu_b |S \nu\); however, \( \mu_b \perp \nu \) and hence \( S \mu_b = 1 \). Thus \( S \mu = S \mu_c \) is cyclic.

**REMARKS.** 1. Corollary 1 may be proved directly using an argument similar to the one used in the proof of Theorem 1.

2. Since Shapiro-Korenblum-Roberts characterization of cyclic inner functions remains valid in the Bergman spaces \( A_\alpha^p \) with \( \alpha > -1 \) and \( 0 < p < \infty \), our results extend readily to these spaces. Here the subscript \( \alpha \) indicates that the Bergman norm is induced by the weighted measure \((1 - r)^\alpha r dr d\theta\).

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**REFERENCES**


