UNORIENTED BRANCHED COVERINGS
ARISING FROM GROUP ACTIONS

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Abstract. For an unbranched covering \( f: M^n \rightarrow N^n \), \([M] = (\text{deg } f)[N]\) in unoriented cobordism \( \mathcal{R}_\ast \). Thus, in general, if \( f: M \rightarrow N \) is a branched covering, then \([M] - (\text{deg } f)[N]\) depends upon the branching behavior.

In this note we describe the ideal \( I_G \) of unoriented cobordism classes \([M^n] - [G][M^n/G]\), where \( G \) is a finite group acting on \( M \) so that \( M \rightarrow M/G \) is a \(|G|\)-fold smooth branched covering of closed smooth manifolds.

1. Introduction. The purpose of this note is to describe the ideal \( I_G \) of unoriented cobordism classes \([M^n] - [G][M^n/G]\), where \( G \) is a finite group acting on \( M \) so that \( M \rightarrow M/G \) is a \(|G|\)-fold smooth branched covering of closed smooth manifolds.

It is known that for an unbranched covering \( f: M^n \rightarrow N^n \), \([M] = (\text{deg } f)[N]\) in unoriented cobordism \( \mathcal{R}_\ast \). Thus, in general, if \( f: M \rightarrow N \) is a branched covering, then \([M] - (\text{deg } f)[N]\) depends upon the branching behavior.

We follow Brand [1] in defining branched covering, including a smoothness condition:

Definition. A branched covering is a smooth map \( f: M^n \rightarrow N^n \) between closed smooth manifolds which is finite-to-one and open. The singular set is the set of points of \( M \) at which \( f \) is not a local homeomorphism, and the branch set \( B_f \) is the image under \( f \) of the singular set. We assume that \( B_f \) is a smoothly embedded codimension 2 submanifold of \( N \).

Let \( B_k \leq 2 \) be a component of \( f^{-1}B_f \). Then, on a neighborhood of \( B_k \), \( f \) is locally isomorphic to the map \( D^{n-2} \times D^2 \rightarrow D^{n-2} \times D^2: (x, z) \rightarrow (x, z^k) \). The number \( k \) is called the local branching degree. For more details, see Stong [5].

Note that \( M - f^{-1}B_f \rightarrow N - B_f \) is an unbranched covering. If \( M \) and \( N \) are path-connected and if \( \pi_1(N - B_f)/\pi_1(M - f^{-1}B_f) \cong G \), a finite group with \(|G| = \text{deg } f\), then \( G \) acts on \( M \), freely on \( M - f^{-1}B_f \), with cyclic isotropy groups, and \( M \rightarrow M/G \) is the branched covering.

Conversely, if \( G \) acts on \( M \) with cyclic isotropy groups such that \( \{ x \in M: G_x \neq \{1\}\} \) is a codimension 2 submanifold of \( M \), then \( M \rightarrow M/G \) is a branched covering. If \( F_k \) is a component of the fixed point set of \( Z_k \subset G \), then \( k \) is the local branching degree on \( F_k \).

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Notation. If $G = \mathbb{Z}_d$, then we write $I_d$ for $I_G$. We write $RP(n_1, \ldots, n_k)$ to mean the real projective space of the bundle $\gamma_{n_1} \oplus \cdots \oplus \gamma_{n_k}$ sitting over $RP^n \times \cdots \times RP^n$, where $\gamma_{n_i}$ is the standard line bundle over $RP^n$ [6].

In §2, we determine $I_G$ in unoriented cobordism $\mathcal{R}_*$ for all finite groups $G$. We begin by noting the following reduction of the problem:

Proposition A. If $G$ is a finite group with a Sylow 2-subgroup $S$, then $I_G = I_S$ in $\mathcal{R}_*$.

Stong [6] shows that $I_2 = \bigoplus_n \{ \alpha \in \mathcal{R}_n : w_1^n(\alpha) = 0 \}$ in $\mathcal{R}_*$.

Proposition B. For $s > 1$, $I_{2^s} = \bigoplus_n \{ \alpha \in \mathcal{R}_n : w_1^n(\alpha) = 0, 1 \leq i \leq n \}$ in $\mathcal{R}_*$.

Proposition C. If $S$ is a 2-group containing $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $I_S = 0$ in $\mathcal{R}_*$. If $S$ is a nonabelian 2-group that does not contain $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $S$ is a generalized quaternion group. Therefore, we finish our study with the following:

Proposition D. If $Q$ is a generalized quaternion group, then $I_Q$ in $\mathcal{R}_*$ is the ideal generated by $[RP(0, m, 2j + 1, 2j + 1)]$, $m \geq 4j + 3$.

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2. Unoriented branched coverings. Many of our results rely upon the following proposition due to Stong [5].

Proposition 2.1. If $f : M^n \rightarrow N^n$ is a branched covering, the $[M] - (\deg f)[N] = [RP(\nu_{\text{even}} + 1)]$ in $\mathcal{R}_*$, where $\nu_{\text{even}} = \bigcup_k n_k$, with $\nu_k$ the normal bundle of $B_k$ in $M$. □

The following reduction of the problem follows from the above result.

Proposition 2.2. If $G$ is a finite group with a Sylow 2-subgroup $S$, then $I_G = I_S$ in $\mathcal{R}_*$.

Proof. If $M^n \rightarrow M^n/G$ is a branched covering, then


$$= [M] - |S|[M/S],$$

and $M \rightarrow M/S$ is a branched covering. Thus, $I_G \subseteq I_S$. (Note: By Proposition 2.1, if $S = \{1\}$, then $[M] - |G|[M/G]$ is 0 because $M \rightarrow M/G$ is a branched covering with only odd local branching degrees.)

If $M \rightarrow M/S$ is a branched covering, then $G$ acts on $G \times M$ as multiplication on $G$. $G \times M \rightarrow G \times M/G \equiv M/S$ is a branched covering with $[G \times M] - |G|[M/S] = [G:S][M] - |G|[M/S] = [G:S][M] - |S|[M/S] = [M] - |S|[M/S].$ Thus $I_S \subseteq I_G$. □

Observation. According to Stong [6], $I_2 = \{ \alpha : w_1^n(\alpha) = 0 \}$ in $\mathcal{R}_*$.

Proposition 2.3. For $s > 1$, $I_{2^s} = \{ \alpha : w_1^n(\alpha) = 0, 0 \leq i \leq n \}$ in $\mathcal{R}_*$. 

Proof. Let $M^n \to M^n/\mathbb{Z}_2$ be a branched covering, and let $F^{n-2} \subset M^n$ be the singular set. $F = A \cup B$, where $A$ is the set of points with isotropy group $\mathbb{Z}_2$ and $B = F - A$.

Let $\nu \to F$ be the normal bundle of $F$ in $M$, and let $c \in H^1(\mathbb{P}(\nu + 1); \mathbb{Z}_2)$ be the first Stiefel-Whitney class of the double cover by the sphere bundle. We will often consider $c$ to be the corresponding class in $H^1(\mathbb{P}(\nu); \mathbb{Z}_2)$ because the classifying map $\mathbb{P}(\nu) \to \mathbb{P}(\nu + 1) \to \mathbb{P}^\infty$. If $w(\nu) = 1 + u_1 + u_2$, then $w(\mathbb{P}(\nu)) = w(F)(1 + c)^2 + u_1(1 + c) + u_2 = w(F)(1 + u_1)$ since $c^2 + cu_1 + u_2 = 0$. Because $\nu|_B \to B$ is a complex vector bundle, $u_1|_B = 0$, implying that $w(\mathbb{P}(\nu)) = w(A)(1 + u_1) + w(B)$.

$\mathbb{Z}_2/\mathbb{Z}_2 = \mathbb{Z}_{2^{n-1}} \neq \{1\}$ acts freely on $A$. $\mathbb{Z}_2$ acts as multiplication by $-1$ in the fibers of $\nu|_A \to A$; thus, $\mathbb{Z}_2$ acts as $\det(-/2) = 1$ on $\det \nu|_A \to A$. It follows that $\mathbb{Z}_2$, which acts freely on $\det \nu|_A \to A$; hence, $\det \nu|_A \to A$ bounds. Because $w(\det \nu|_A) = 1 + u_1$, $u_1[w_1(A)[A]] = 0$. In particular, $0 = \{w(A)(1 + u_1)\}w[A] = w(B)[B]$ since according to Stong [5], $\mathbb{P}(\nu) \to \mathbb{P}^\infty$ bounds.

$$w(\mathbb{P}(\nu + 1)) = w(F)\left(1 + (c + u_1)(1 + c)^2 + u_2(1 + c)\right) = w(F)\left(1 + (c + u_1 + (c^2 + u_2)\right),$$

since $c^3 + u_1c^2 + u_2c = 0$. Recall that $[\mathbb{P}(\nu + 1)] = [M]$ in $\mathbb{Z}_2$.

$$w_n[\mathbb{P}(\nu + 1)] = w_{n-2}(F)(c^2 + u_2)[\mathbb{P}(\nu + 1)] = w_{n-2}(F)[F] = 0,$$

$$w_1w_{n-1}[\mathbb{P}(\nu + 1)] = (w_1(F) + c + u_1)(w_{n-2}(F)(c + u_1)$$

$$+ w_{n-3}(F)(c^2 + u_2))[\mathbb{P}(\nu + 1)] = (2u_1w_{n-3}(F) + w_{n-2}(F) + w_{n-3}(F)w_1(F))[F] = 0.$$

We know that $w^n[M] = 0$, so assume $2 \leq i \leq n - 2$.

$$w_1w_{n-i}[\mathbb{P}(\nu + 1)] = (w_1(F) + c + u_1)^i(w_{n-i}(F) + w_{n-i-1}(F)(c + u_1)$$

$$+ w_{n-i-2}(F)(c^2 + u_2)) = \sum_{j=0}^{i} \binom{i}{j} c^j(w_1(F) + u_1)^{i-j}(w_{n-1}(F) + w_{n-i-1}(F)u_1 + w_{n-i-2}(F)u_2$$

$$+ \sum_{j=0}^{i} \binom{i}{j} c^j(w_1(F) + u_1)^{i-j}(w_{n-i-1}(F)c + w_{n-i-2}(F)c^2).$$

Now,

$$c^j(w_1(F) + u_1)^{i-j}(w_{n-i}(F) + w_{n-i-1}(F)u_1)[\mathbb{P}(\nu + 1)]$$

$$= \bar{u}_{n-2}w_1[\mathbb{P}(\nu)]^{i-j}w_{n-i}[\mathbb{P}(\nu)][F] = 0,$$

where $\bar{u} = 1/(1 + u_1 + u_2)$. 


\[ w_i w_{n-i} [R \mathbb{P}(\nu + 1)] = \sum_{j=2}^{i} \binom{i}{j} c^{i-j} (w_1(F) + u_1) w_{n-i-2}(F)(c^3 + c^2u_1)[R \mathbb{P}(\nu + 1)] \]

\[ + \sum_{j=0}^{i} \binom{i}{j} c^j (w_1(F) + u_1) w_{n-i-1}(F)c + w_{n-i-2}(F)c^2)[R \mathbb{P}(\nu + 1)] \]

\[ = \sum_{j=2}^{i} c^{i-j} (w_1(F) + u_1) w_{n-i-2}(F)u_1 + w_{n-i-1}(F))[R \mathbb{P}(\nu + 1)] \]

\[ + \sum_{j=0}^{i} \binom{i}{j} c^j (w_1(F) + u_1) w_{n-i-1}(F)c + w_{n-i-2}(F)c^2)[R \mathbb{P}(\nu + 1)] \]

\[ = \{ (w_1(F) + u_1) w_{n-i-2}(F) + i(w_1(F) + u_1) w_{n-i-1}(R \mathbb{P}(\nu)) \}[F] \]

\[ = (w_1(A) + u_1) w_{n-i-2}(A)[A] + w_1(B) w_{n-i-2}(B)[B] = 0. \]

Thus, \( w_i w_{n-i}[M] = 0 \) for \( 0 \leq i \leq n \).

Set \( J = \{ \alpha : w_i w_{n-i}(\alpha) = 0, 0 \leq i \leq n \} \). According to Capobianco [3], \( \alpha \in J \) can be represented by a fibering with fiber \( R \mathbb{P}(3) \), and Stong [6] exhibits generators of \( J \); they are of the form \( R \mathbb{P}(\xi + \eta) \), where \( \xi \) is a complex line bundle and \( \eta \) is a real 2-plane bundle. There is a \( \mathbb{Z}_2 \) action on \( R \mathbb{P}(\xi + \eta) \) induced by multiplication by the \( 2^\text{nd} \) roots of 1 in the fibers of \( \xi \), and by 1 in the fibers of \( \eta \). The fixed set is \( R \mathbb{P}(\xi) \cup R \mathbb{P}(\eta) \), which has codimension 2; thus, \( J = I_2 \). \( \Box \)

**Proposition 2.4.** If \( S \) is a 2-group containing \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( I_S = 0 \) in \( \mathfrak{M}_* \).

**Proof.** If \( M \rightarrow M/S \) is a branched covering, then the isotropy groups of the \( S \) action on \( M \) are cyclic. In particular, \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts without stationary points, implying \([M] = 0 \) in \( \mathfrak{M}_* \) [4, 30.1]. \( \Box \)

**Remark.** If \( S \) is a nonabelian 2-group that does not contain \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( S \) is a generalized quaternion group.

Let \( Q_8 = \{ \pm 1, \pm i, \pm j, \pm ij \} \) be the quaternion group, let \( M^n \rightarrow M^n/Q_8 \) be a branched covering and let \( F^{n-2} \) be the fixed point set. Then, \( F = A_2 \cup B_i \cup C_j \cup D_{ij} \), where \( A_2 \) is the fixed point set of \( Z_2 \) in \( Q_8 \), \( B_i \) is the fixed point set of \( \langle i \rangle \) in \( Q_8 \), etc.

\( \langle j \rangle \) gives us a \( Z_4 \) action on the normal bundle of \( B_j, \nu \rightarrow B \), such that \( Z_2 \subset Z_4 \) acts trivially on \( B \) and as multiplication by \(-1 \) in the fibers of \( \nu \). Also, \( Z_4/Z_2 = Z_2 \) acts freely on \( B \).

Beem [2] supplies us with a classifying space for such bundles: Write \( C^\infty = R^\infty + jR^\infty \). Then \( j \) acts on \( C^\infty \) by multiplication. There is an induced action on \( \gamma^2 \rightarrow BO_2(C^\infty) \) such that \( Z_2 \subset \langle j \rangle \) acts trivially on \( BO(C^\infty) \) and as multiplication by \(-1 \) in the fibers of \( \gamma^2 \). We can classify a free \( Z_2 \) action by a map into \([S^\infty, -1] \). Thus, the desired classifying space is \( BO_2(C^\infty) \times S^\infty/Z_2 \). Beem [2] determines that

\[ H^*(BO_2(C^\infty) \times S^\infty/Z_2; Z_2) = Z_2[w_1, w_2^2, \alpha]/\langle \alpha^2 w_1 \rangle, \]
where \( w_1, w_2 \) are the universal classes which generate \( H^*(BO_2; \mathbb{Z}_2) \) and \( \alpha \) generates \( H^*(RP^\infty; \mathbb{Z}_2) \).

We have the following commutative diagram:

\[
\begin{array}{ccc}
B & \rightarrow & BO_2(C^\infty) \times S^\infty \\
\downarrow & & \downarrow \\
B/\mathbb{Z}_2 & \rightarrow & BO_2(C^\infty) \times S^\infty/\mathbb{Z}_2
\end{array}
\]

The classes \( w_1 \) and \( w_2 \) pull back to \( u_1 \) and \( u_2 \) in \( H^*(B; \mathbb{Z}_2) \), where \( w(v) = 1 + u_1 + u_2 \). Note that \( u_1 = 0 \) because \( v \rightarrow B \) is a complex vector bundle. Thus, \( w_\omega u_2^j[B] = 0 \) for all \( \omega, j \) since these numbers come from \( B/\mathbb{Z}_2 \). We obtain the same result for \( A, C, \) and \( D \).

In fact, for \( A \) we can say more. There is a \( Q_8 \) action on \( \gamma^2 \rightarrow BO_2(H^\infty) \) induced by multiplication such that \( \mathbb{Z}_2 \subset Q_8 \) acts trivially on \( BO_2(H^\infty) \) and as multiplication by \(-1\) in the fibers of \( \gamma^2 \). We have such an action on \( \nu \rightarrow A \); also, \( Q_8/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts freely on \( A \). Such an action can be classified by a map into \([S^\infty \times S^\infty; -1 \times 1, 1 \times -1]\). Thus, we have a classifying space for \( A/\mathbb{Z}_2 \times \mathbb{Z}_2 \), namely \( BO_2(H^\infty) \times S^\infty \times S^\infty/\mathbb{Z}_2 \times \mathbb{Z}_2 \) = \( \mathbb{Z}_2[\alpha, \beta, w_1, w_2] \)/relations.

From the double covers \( A \rightarrow A/\mathbb{Z}_2 \) and \( BO_2(C^\infty) \times S^\infty \rightarrow BO_2(C^\infty) \times S^\infty/\mathbb{Z}_2 \), we have the following exact sequences:

\[
\begin{array}{ccc}
H^*(A; \mathbb{Z}_2) & \leftarrow & H^*(A/\mathbb{Z}_2; \mathbb{Z}_2) & \leftarrow & H^*(A \times I/\mathbb{Z}_2 \times -1; \mathbb{Z}_2) \\
\text{up} & & \text{up} & & \text{up} \\
H^*(BO_2 \times S^\infty; \mathbb{Z}_2) & \leftarrow & H^*(BO_2 \times S^\infty/\mathbb{Z}_2; \mathbb{Z}_2) & \leftarrow & H^*(BO_2 \times S^\infty \times I/\mathbb{Z}_2 \times -1; \mathbb{Z}_2) \\
\text{up} & & \text{up} & & \text{up} \\
\mathbb{Z}_2[w_1, w_2] & \leftarrow & \mathbb{Z}_2[w_1, w_2, \alpha]/\langle \alpha^2 w_1 \rangle & \leftarrow & \langle \mathbb{Z}_2[w_1, w_2, \alpha]/\langle \alpha^2 w_1 \rangle, U \rangle
\end{array}
\]

where \( U \) is the Thom class of the line bundle \( BO_2(C^\infty) \times S^\infty \times I/\mathbb{Z}_2 \times -1 \rightarrow BO_2(C^\infty) \times S^\infty/\mathbb{Z}_2 \).

\[
\langle w_\omega u_1^j u_2^{j+1}, [A] \rangle = \langle w_\omega u_1^j u_2^{j+1}, \delta [A \times I/\mathbb{Z}_2 \times -1] \rangle = \langle \delta (w_\omega u_1^j u_2^{j+1}), [A \times I, \mathbb{Z}_2 \times -1] \rangle = \langle w_\omega u_1^j u_2^{j}(\delta u_2), [A \times I, \mathbb{Z}_2 \times -1] \rangle.
\]

In the cohomology of the universal space, \( \delta w_2 \) is a class such that \( \alpha(\delta w_2) = 0 \); thus, \( \delta w_2 = \omega w_2 U, \) implying

\[
\langle w_\omega u_1^j u_2^{j+1}, [A] \rangle = \langle w_\omega u_1^j u_2^{j} \omega u_1 U, [A \times I/\mathbb{Z}_2 \times -1] \rangle = \langle w_\omega u_1^{j+1} u_2^j \alpha, [A/\mathbb{Z}_2] \rangle = 0
\]

since \( u_2^j \) comes from \( H^*(A/\mathbb{Z}_2 \times \mathbb{Z}_2; \mathbb{Z}_2) \).

We consider equivariant bordism of our special \( Q_8 \) actions, that is, \( Q_8 \) actions such that each action is free off of a codimension 2 submanifold, and on that
submanifold the isotropy groups are nontrivial cyclic groups. We obtain Conner-
Floyd exact sequences:

\[
\begin{align*}
\mathcal{R}_\ast(Q_8) & \rightarrow \mathcal{R}_\ast(Q_8, \text{Free } \partial) \rightarrow \mathcal{R}_{\ast-1} \text{(Free } Q_8) \\
& \downarrow \text{transfer} \\
\mathcal{R}_\ast(Z_2) & \rightarrow \mathcal{R}_\ast(Z_2, \text{Free } \partial) \rightarrow \mathcal{R}_{\ast-1} \text{(Free } Z_2)
\end{align*}
\]

where the vertical arrows are obtained by restricting to the \( Z_2 \subset Q_8 \) action. Classification of the normal bundle to the fixed set leads to

\[
\mathcal{R}_\ast(Q_8, \text{Free } \partial) = \mathcal{R}_{\ast-2}(BO_2(H^\infty) \times S^\infty \times S^\infty / Z_2 \times Z_2)
\]

\[+ 3 \mathcal{R}_{\ast-2}(BU_1(H^\infty) \times S^\infty / Z_2).\]

and \( \mathcal{R}_\ast(Z_2, \text{Free } \partial) = \mathcal{R}_{\ast-2}(BO_2(H^\infty)). \) A class in \( \mathcal{R}_{\ast-2}(BO_2) \) is determined by the numbers \( w_* u_1 u_2^2[F] \); in fact, the numbers \( w_* u_1^2 u_2^2[A] \) determine a class in the image of transfer: Since \( RP(v) \rightarrow RP^\infty \) bounds,

\[0 = c^{4j+3} w_\ast \left[ RP(v) \right] = \bar{u}_{4j+2} \left\{ (1 + u_1) w(A) \right\} \omega \left[ A \right] + u_{2j+1}^2 w_\ast \left[ B \cup C \cup D \right].\]

**Lemma 2.5.** The image of the transfer map

\[\text{tr}: \mathcal{R}_\ast(BO_2(H^\infty) \times S^\infty \times S^\infty / Z_2 \times Z_2) \rightarrow \mathcal{R}_\ast(BO_2(H^\infty))\]

is a free \( \mathcal{R}_\ast \)-module with a basis in the image of the homology transfer.

**Proof.** Let \( \pi: \tilde{X} \rightarrow X \) be a double cover. There is a line bundle \( L \) associated to this cover with \( EL = \tilde{X} \times \mathbb{R} / Z_2 \times -1 \). Let \( [M, f] \in \mathcal{R}_\ast(X) \); then, we have a double cover of \( M, \tilde{M} = \{(m, \tilde{x}) \in M \times \tilde{X} : f(m) = \pi(\tilde{x}) \} \). The Thom isomorphism \( T: \mathcal{R}_\ast(X) = \pi^\ast_+(T) = \pi^\ast_+(DL, SI) \) sends \( [M] \) to \( [\tilde{M} \times I / Z_2 \times -1, \tilde{M} \times \{ \pm 1 \} / Z_2 \times -1] \), and the boundary homomorphism \( \partial: \pi^\ast_+(DL, SI) \rightarrow \pi^\ast_+(SI) = \mathcal{R}_\ast(X) \) sends the given pair to \( [\tilde{M}] \). The composite \( \partial T: \mathcal{R}_\ast(X) \rightarrow \mathcal{R}_\ast(\tilde{X}) \) is just the transfer map associated to the double cover \( \tilde{X} \rightarrow X \).

If \( f: Y \rightarrow Z \) is a map, then the image of \( f_*: \mathcal{R}_\ast(Y) \rightarrow \mathcal{R}_\ast(Z) \) is a free \( \mathcal{R}_\ast \)-module with a basis in the image of \( f_*: H_\ast(Y) \rightarrow H_\ast(Z) \). Now \( \partial \) is the composite of \( i_*: \pi^\ast_+(DL, SI) \rightarrow \pi^\ast_+(\Sigma SI) \), where \( i \) is the inclusion, and the suspension isomorphism \( \sigma: \pi^\ast_+(\Sigma SI) \rightarrow \pi^\ast_+(SI) \). Therefore, the image of \( \partial T: \mathcal{R}_\ast(X) \rightarrow \mathcal{R}_\ast(\tilde{X}) \) is a free \( \mathcal{R}_\ast \)-module with a basis in the image of the homology transfer. It follows that this result holds for the transfer map associated to the double-double cover, \( BO_2(H^\infty) \times S^\infty \times S^\infty \rightarrow BO_2(H^\infty) \times S^\infty \times S^\infty / Z_2 \times Z_2 \). \( \square \)

We now need to find a basis for the image of the homology transfer. Consider the fibering \( RP^3 \rightarrow RP(\gamma_{2j+1}^n \oplus \gamma^1_n \oplus \gamma^1_m) \rightarrow CP^{2j+1} \times RP^n \times RP^m \). \( NS^1 \) in \( S^3 \) acts on \( \gamma_{2j+1}^n \rightarrow CP^{2j+1} \) by multiplication, and it acts on \( \gamma^1_n + \gamma^1_m \rightarrow RP^n \times RP^m \) trivially. In the induced action on \( RP(\gamma_{2j+1}^n + \gamma^1_n + \gamma^1_m) \), \( S^1 \) fixes \( RP(\gamma^1_n + \gamma^1_m) \), and \( Z_2 \subset S^1 \) fixes \( RP(\gamma_{2j+1}) \). We have \( Q_8 \subset NS^1 \) acts on \( RP(\gamma_{2j+1} + \gamma^1_n + \gamma^1_m) \) that induces a branched covering. The normal bundle of \( RP(\gamma_{2j+1}) \) in \( RP(\gamma_{2j+1} + \gamma^1_n + \gamma^1_m) \) is
\[ v = \mu \otimes (\gamma_n \oplus \gamma_m) \], where \( \mu \to \mathbb{R}^P^{4j+3} \times \mathbb{R}^P^n \times \mathbb{R}^P^m \) is the pullback of the standard line bundle over \( \mathbb{R}^P^{4j+3} \). Then,

\[ w(v) = (1 + c + a)(1 + c + b) = 1 + (a + b) + (c^2 + c(a + b) + ab) \]

with \( a \in H^1(\mathbb{R}^P^n; \mathbb{Z}_2), b \in H^1(\mathbb{R}^P^m; \mathbb{Z}_2), \) and \( c \in H^1(\mathbb{R}^P^{4j+3}; \mathbb{Z}_2), \) the nonzero classes.

Let \( n = 0 \); then, \( w(v) = 1 + b + c(c + b) \), and \( u^*_1u^{4j+3}_2 = b'[c^{4j+3}(c + b)^{4j+3}] \)

\[ u^{n-(4j+3)}_1u^{4j+3}_2 \left[ \mathbb{R}P(\mathbb{Z}_2,1) \right] = \mathbb{S}^{4j+3}b^m[\mathbb{R}P^{4j+3} \times \mathbb{R}P^m] = 1. \]

Since these numbers are the only possible nonzero numbers of the form \( u^*_1u^*_2[A] \), we get all possible nonzero numbers \( u^*_1u^*_2[A] \) by taking

\[ [\mathbb{M}] = [\mathbb{R}P(0, m, 2j + 1, 2j + 1)], \quad \text{with } m \geq 4j + 3. \]

Because \( H^*(BO_2; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2] \), it follows that the homology classes corresponding to the \( [\mathbb{R}P(\mathbb{Z}_2,1)] \) form a basis for the image of \( \text{tr}: H_*(BO_2(\mathbb{H}^\infty); \mathbb{Z}_2) \rightarrow H_*(BO_2(\mathbb{H}^\infty); \mathbb{Z}_2). \)

**Proposition 2.6.** \( I_{Q_n} \) in \( \mathcal{N}_* \) is the ideal \( I \) generated by \( [\mathbb{R}P(0, m, 2j + 1, 2j + 1)], \) with \( m \geq 4j + 3. \)

**Proof.** The ideal \( I \) is certainly contained in \( I_Q \). If \([M] \in \mathcal{I}_Q, \) then \([M, \mathbb{Z}_2] \in \mathcal{N}_*(\mathbb{Z}_2) \) is determined by \([F, \nu] \in \mathcal{N}_{*-2}(BO_2) \), which in turn is determined by \([A, \nu] \in \mathcal{N}_{*-2}(BO_2) \). By Lemma 2.5, the collection of all such \([A, \nu] \) is a free \( \mathcal{N}_* \)-module on a basis given by the homology images of \( \nu: \mathbb{R}P(\mathbb{Z}_2,1) \subseteq \mathbb{R}P(0, m, 2j + 1, 2j + 1), m \geq 4j + 3. \) Thus, \( I_Q = I. \)

Since the above arguments can be used for branchings which come from actions by a generalized quaternion group, we have the following

**Corollary 2.7.** \( I_{Q_n}, r \geq 3, \) is the ideal in \( \mathcal{N}_* \) generated by

\[ [\mathbb{R}P(0, m, 2j + 1, 2j + 1)], \quad m \geq 4j + 3. \]

**Remark.** If \( Q_8 \) acts on \( M^n \) so that \( M \to M/Q \) is a branched covering, then \( w_1w_2w_3w_4[M^n] = 0, \) where \( s + i + j + k = n. \) These numbers distinguish classes in \( \mathcal{N}_* \), through dimension \( 7; \) thus, \( (I_Q)_n = 0 \) for \( n \leq 7. \) In fact, \( (I_Q)_8 = 0 \) because \( \mathbb{S}P(0, 3, 1, 1) \) bounds.

We find an indecomposable \( [\mathbb{R}P(2j + 1, 2j + 1, m)], m \geq 4j + 3, \) in dimension \( n \geq 9 \) if \( n \) is not of the form \( 2^s - 3, 2^s - 2, 2^t - 1, \) or \( 2^t. \) Also, if \( n \) is of this form, then \( [\mathbb{R}P(2j + 1, 2j + 1, n_1, n_2)] \), \( 4j + 5 + n_1 + n_2 = n, \) is decomposable.

**References**


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