GORENSTEIN ALGEBRAS AND THE CAYLEY-BACHARACH THEOREM

E. D. DAVIS, A. V. GERAMITA AND F. ORECCHIA

Abstract. This paper is an examination of the connection between the classical Cayley-Bacharach theorem for complete intersections in \( \mathbb{P}^2 \) and properties of graded Gorenstein algebras.

Introduction. It is known, if not well known, that the Cayley-Bacharach theorem for complete intersections in \( \mathbb{P}^2 \) is valid for 0-dimensional arithmetically Gorenstein subschemes of \( \mathbb{P}^n \). More generally, we show that the result is valid for 0-dimensional subschemes of \( \mathbb{P}^n \) having minimal Cohen-Macaulay type compatible with their Hilbert functions. The Cayley-Bacharach theorem in the Gorenstein case is a special instance of a theorem, interesting and technically useful in its own right, relating the Hilbert functions of linked subschemes of \( \mathbb{P}^n \). Lastly we show that the 0-dimensional, arithmetically Gorenstein, reduced subschemes of \( \mathbb{P}^n \) are characterized by the validity of the Cayley-Bacharach theorem and the symmetry of the Hilbert function.

Fixed notation. \( A \) denotes a standard \( \mathbb{N} \)-graded \( k \)-algebra, \( k \) a field: \( A_0 = k \); \( A = k[A] \); \( \lambda(A) < \infty \). We use \( \lambda \) to denote \( k \)-linear dimension, reserving “dim” for dimension of rings or schemes, and \( \delta \) to denote multiplicity for such algebras (or degree of the corresponding projective scheme). Note that \( \delta(A) = \lambda(A) \) if \( \dim A = 0 \). We use \( I \) to denote a nonzero, nonunit, homogeneous ideal of \( A \), and \( J = \text{ann} I \) (“ann” = annihilator). We assume always that \( A \) and \( A/I \) are CM (Cohen-Macaulay) and that \( \text{Ass}(A/I) \subset \text{Ass}(A) \). Hence \( \text{Ass}(A/J) \subset \text{Ass}(A) \). (Indeed, since the 0-ideal of \( A \) is unmixed of height 0, so is the annihilator of any nonzero ideal of \( A \).) Therefore \( A/J \) is CM if \( \dim A \leq 1 \). In any case, \( A/J \) is CM if \( A \) is Gorenstein [PS, Proposition 1.3]. In our applications \( A/J \) will be CM for one of these two reasons.

Recall that, by definition, a ring \( R \) is Gorenstein provided that \( R_\mathfrak{p} \) is a Gorenstein local ring for every prime ideal \( \mathfrak{p} \) of \( R \), and \( A \) is Gorenstein \( \iff A_{A_\mathfrak{p}A} \) is a Gorenstein local ring [AG]. We refer to [K] for those properties of Gorenstein local rings which are used below without specific reference.

1. Observations. Assume that \( A_\mathfrak{p} \) is Gorenstein for all \( \mathfrak{p} \in \text{Ass}(A) \). Then:

(a) \( \delta(A) = \delta(A/I) + \delta(A/J) \).
(b) Suppose \( \dim A = 0 \), and let \( N = \max \{ t \in \mathbb{N} | A_t \neq 0 \} \). Then
\[
\lambda(A_t) = \lambda(I_t) + \lambda(J_{N-t}) = \lambda(A_{N-t}), \quad 0 \leq t \leq N.
\]

**Proof.** (a) Since \( A \) is a 0-dimensional Gorenstein local ring for \( t \in \text{Ass}(A) \),
\[
\text{length}(A_t) = \text{length}(A_t/IA_t) + \text{length}(A_t/JA_t).
\]
Now use the multiplicity formula [N, p. 76]:
\[
\delta(A) = \sum \delta(A_t) \text{length}(A_t) = \sum \delta(A_t) \text{length}(A_t/IA_t) + \sum \delta(A_t) \text{length}(A_t/JA_t)
\]
\[
= \delta(A/I) + \delta(A/J).
\]

(b) Since \( A \) is a 0-dimensional Gorenstein local ring, \( \lambda(\text{ann}(A_t)) = 1 \). Hence
\( \text{ann}(A_1) = A_N \). It follows easily that the \( k \)-bilinear map \( A_i \times A_j \to A_{i+j} \), induced by
multiplication, is nonsingular for \( i + j \leq N \). From this one deduces that
\( \text{ann}(I_t)_{N-t} = J_{N-t} \). (b) now follows easily from the nonsingular \( k \)-bilinear pairing
\( A_i \times A_{N-i} \to A_N \equiv k \).

Observation 1(b), a well-known property of quasi-Frobenius algebras, contains
our theorem relating the Hilbert functions of linked projective schemes. To see this
we require certain standard technicalities.

**Further notation.** \( H(S, -) \) denotes the Hilbert function of the standard \( \mathbb{N} \)-graded
\( k \)-algebra \( S \) (i.e., \( H(S, t) = \lambda(S_t) \)), and \( \Delta \) denotes the difference operator on \( \mathbb{Z} \)-valued
sequences (i.e., if \( f \) is a \( \mathbb{Z} \)-valued sequence, then \( \Delta f(0) = f(0) \) and \( \Delta f(i) = f(i) - f(i-1) \) for \( i > 0 \)). Since \( H(S, t) \) is a degree \( S \) \( 1 \) polynomial function of \( t \) for
\( t \gg 0 \), \( \Delta^{\dim S} H(S, t) = 0 \) for \( t \gg 0 \). Define:
\[
\sigma(S) = 1 + \max \{ t \in \mathbb{N} | \Delta^{\dim S} H(S, t) \neq 0 \}. \]

Observe that for any \( t \geq \sigma(S) - 1 \), \( \delta(S) = \sum (\Delta^{\dim S} H(S, j)) | 0 \leq j \leq t \}, \) and \( \delta(S) = \Delta^{\dim S} H(S, t) \) if \( \dim S > 0 \). For any nonzero, nonunit, homogeneous ideal \( Q \) of \( S \),
define:
\[
\sigma(Q) = \sigma(S/Q); \quad \alpha(Q) = \min \{ t \in \mathbb{N} | Q_t \neq 0 \}. \]

Observe that for any \( m \in \mathbb{N}, \alpha(Q) = \min \{ t \in \mathbb{N} | \Delta^m H(S/Q, t) \neq \Delta^m H(S, t) \}. \)

**Reduction to dimension 0.** Henceforth, for technical convenience, we assume \( k \) to be infinite, in which case there is an \( A \)-primary ideal \( Q \) generated by an \( A \)-regular
sequence in \( A \). (So this sequence is also \( (A/I) \)-regular and, if \( A/J \) is CM, then
\( (A/J) \)-regular.) Let \( x \to \bar{x} \) denote the canonical map \( A \to A/Q = \bar{A} \). By [G],
\( \lambda(\text{ann}((A)) \) is independent of the choice of \( Q \); this integer is called the CM-type of \( A \).
Recall that \( A \) is Gorenstein \( \iff \bar{A} \) is Gorenstein \( \iff \text{CM-type of } A = 1 \).

2. **Observations.** Let \( m = \dim A. \)
(a) \( \Delta^m H(A, -) = H(\bar{A}, -); \Delta^m H(A/I, -) = H(\bar{A}/I, -); \bar{I} \neq 0 \).
(b) \( \delta(A) = \delta(\bar{A}) = \lambda(\bar{A}); \delta(A/I) = \delta(\bar{A}/I) = \lambda(\bar{A}/I). \)
(c) \( \sigma(A) = \sigma(\bar{A}); \sigma(I) = \sigma(\bar{I}); \alpha(I) = \alpha(\bar{I}). \)
(d) \( \alpha(\bar{I}) \leq \sigma(\bar{I}) \leq \sigma(\bar{A}) \neq \alpha(\bar{I}). \)
Proof. (a) follows immediately from the fact that \( Q \) is generated by a sequence which is both \( \mathcal{A} \)- and \( (\mathcal{A}/I) \)-regular, and (b)–(d) follow formally from (a) and definitions.

3. **Theorem (Hilbert Functions under Liaison).** Suppose \( A \) is Gorenstein. Let \( m = \dim A, N = \sigma(A) - 1 \). Then:

(a) \( \Delta^m H(A, t) = \Delta^m H(A, N - t), 0 \leq t \leq N. \)
(b) \( \Delta^m H(A, t) = \Delta^m H(A/I, t) + \Delta^m H(A/J, N - t), 0 \leq t \leq N. \)
(c) \( \alpha(I) + \alpha(J) = \alpha(J) + \alpha(I) = \sigma(A). \)

Proof. First note that (c) is a formal consequence of (b) and definitions, and that (a) and (b) follow immediately from 1(b) if \( m = 0 \). Hence (a) and (b) follow immediately from 2(a, c), the 0-dimensional case, and

**Claim.** \( J = \text{ann } \mathcal{I}. \)

**Proof.** By 2(b) and 1(a)

\[
\lambda(\overline{A}/J) = \delta(A/J) = \delta(A) - \delta(A/I) = \delta(\overline{A}) - \delta(\overline{A}/\mathcal{I})
\]

\[
= \delta(\overline{A}/\text{ann } \mathcal{I}) = \lambda(\overline{A}/\text{ann } \mathcal{I}).
\]

Since \( \mathcal{J} \subseteq \text{ann } \mathcal{I}, \mathcal{J} = \text{ann } \mathcal{I} \), and we are done.

**Remark.** Suppose \( A \) is Gorenstein. Then, as a corollary to 3(c), we have the validity of the Cayley-Bacharach theorem for \( A \):

\[
\delta(A/I) = \delta(A) - 1 \Rightarrow \alpha(I) = \sigma(A) - 1.
\]

(Proof. \( \delta(A/J) = 1 \), whence \( \sigma(J) = 1 \).) We call this result "Cayley-Bacharach" because: specializing to the case in which \( \text{Proj}(A) \) is the complete intersection of two curves in \( \mathbb{P}^2 \), in which case \( \sigma(A) \) is one less than the sum of the degrees of the curves, we obtain the classical Cayley-Bacharach theorem. (See [SR, pp. 97–101] for further details.) More generally, 3(c) gives: \( \text{Proj}(A/I) \) is in generic position in \( \text{Proj}(A) \Leftrightarrow \text{Proj}(A/J) \) is in generic position in \( \text{Proj}(A). \) ("Generic position" simply means "\( \sigma \leq \sigma + 1 \);" see [O] for a geometric interpretation of "generic position in \( \mathbb{P}^n \)"). The validity of Cayley-Bacharach for \( A \) is, in fact, a consequence of only "half" of the Gorenstein property, since more generally we have

4. **Theorem.** \( \Delta^{\dim A} H(A, \sigma(A) - 1) \leq \text{CM-type of } A. \) If equality holds, then

\[
\delta(A/I) \leq \delta(A) - \sigma(A) + \alpha(I) \leq \delta(A) - 1.
\]

**Proof (cf. [DM, proof of (2.3)]).** CM-type of \( A = \lambda(\text{ann } \overline{A}_1) \geq \lambda(\overline{A}_N) = \Delta^{\dim A} H(A, N) \) \( (N = \sigma(A) - 1) \); equality \( \Leftrightarrow \text{ann } \overline{A}_1 = \overline{A}_N \). If equality holds, then \( \mathcal{I} \neq 0 (\alpha(\mathcal{I}) \leq t \leq N) \), whence, using 2(b, c, d):

\[
\delta(A/I) = \lambda(\overline{A}/\mathcal{I}) \leq \lambda(\overline{A}) - (\sigma(\overline{A}) - \alpha(\mathcal{I}))
\]

\[
= \delta(A) - \sigma(A) + \alpha(I) \leq \delta(A) - 1.
\]

**Remark.** Observe that \( \delta(A/I) = \delta(A) - 1 \Rightarrow J \in \text{Ass}(A) \) and \( \delta(A/J) = 1 \), i.e., \( \text{Proj}(A/J) \) is a linear component of \( \text{Proj}(A)_{\text{red}} \). The existence of such a component is guaranteed if and only if \( \text{Proj}(A) \) is 0-dimensional and has a \( k \)-rational point. That
is, the natural domain of applicability of "Cayley-Bacharach" is that of 0-dimen-
sional subschemes of $\mathbb{P}^n(k = \overline{k})$. Although such schemes may have "Cayley-
Bacharach" without being arithmetically Gorenstein, we have

5. **Theorem.** Suppose that $A$ is reduced, $\dim A = 1$, and every point of $\text{Proj}(A)$ is
$k$-rational. Then $A$ is Gorenstein if and only if the following two conditions are satisfied.
(Let $N = \sigma(A) - 1$.)

(a) (Symmetric Hilbert function)
$$\Delta H(A, t) = \Delta H(A, N - t), \quad 0 \leq t \leq N.$$  

(b) (Cayley-Bacharach)
$$\alpha(\text{ann } \mathfrak{g}) = N \text{ for all } \mathfrak{g} \in \text{Ass}(A).$$

**Proof.** In view of 3, we need only prove the sufficiency of (a) and (b). Let $C$ be
the conductor of $A$ in its integral closure $B$. We shall prove that $A_{A,A}$ is Gorenstein
by verifying that $\lambda(B/C) = 2\lambda(A/C)$ [HK, p. 32].

Identify $B$ as an $A$-algebra and an $N$-graded $k$-algebra with $\bigoplus \{ A/\mathfrak{g} | \mathfrak{g} \in \text{Ass}(A) \}$. Note that $A/\mathfrak{g} \cong k[T]$ (graded $k$-algebra isomorphism). Under these circumstances
$C$ is the ideal in $A$ (and in $B$), $\Sigma\{ \text{ann } \mathfrak{g} | \mathfrak{g} \in \text{Ass}(A) \}$ [O, Proposition 2.5]. So, by (b),
$C_j = 0$ for $0 \leq j < N$ and $\lambda(C_N) = \delta = \delta(A) = \text{card}(\text{Ass}(A))$. On the other hand,
$\lambda(B_t) = \delta$ $(t > 0)$ and $\lambda(A_N) = H(A, N) = \delta$. Consequently, $A_N = B_N = C_N,
\lambda(B/C) = N\delta$ and $\lambda(A/C) = \Sigma\{ H(A, t) | 0 \leq t \leq N - 1 \}$. Now, $H(A, t) = \Sigma\{ \Delta H(A, j) | 0 \leq j \leq t \} = H(A, N) - \Sigma\{ \Delta H(A, j) | t + 1 \leq j \leq N \}$. Then, using
(a),
$$H(A, t) = \delta - \Sigma\{ \Delta H(A, j) | 0 \leq j \leq N - 1 - t \} = \delta - H(A, N - 1 - t).$$

Consequently, $2\lambda(A/C) = 2\Sigma\{ H(A, t) | 0 \leq t \leq N - 1 \} = N\delta = \lambda(B/C)$.

**Remarks.** We do not know to what extent the hypothesis "reduced" can be
eliminated from 5. In case $\lambda(A_1) \leq 3$, i.e., in case $\text{Proj}(A)$ is a subscheme of $\mathbb{P}^2$, [DM]
proves 5 without "reduced", and a stronger result than 5 with "reduced". That
analysis depends heavily on the fact that, in $\mathbb{P}^2$, "arithmetically Gorenstein" = "complete
intersection". 5 should also be compared with Stanley's characterization of
Gorenstein domains among the CM domains [S, Theorem 4.4].

**References**

(1975), 19–23.


Department of Mathematics, SUNY, Center at Albany, Albany, New York 12222

Department of Mathematics, Queen’s University, Kingston K7L 3N6, Ontario, Canada

Department of Mathematics, Università di Genova, Genova, Italy