THE CONVEXITY OF A DOMAIN AND THE SUPERHARMONICITY OF THE SIGNED DISTANCE FUNCTION

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Abstract. Let $D$ be a domain in $\mathbb{R}^N$ with nonempty boundary $\partial D$ and let $u$ be the signed distance function from $\partial D$, i.e. $u = \pm \text{dist}$ according as we are in or outside $\overline{D}$. We prove that, for any $N \geq 2$, $u$ is superharmonic in $\mathbb{R}^N$ if and only if $D$ is convex. When $N = 2$, this criterion requires the superharmonicity of $u$ in $D$ only.

1. Throughout this paper $D$ will denote a proper subdomain of the Euclidean space $\mathbb{R}^N$, where $N \geq 2$. Thus the boundary $\partial D$ of $D$ in $\mathbb{R}^N$ is not empty and we can define the distance function $d$ from $\partial D$. The signed distance function $u$ in $\mathbb{R}^N$ is defined by

$$u = \begin{cases} d & \text{in } \overline{D}, \\ -d & \text{in } D^\prime, \end{cases}$$

where $\overline{D}$ is the closure of $D$ in $\mathbb{R}^N$ and $D^\prime = \mathbb{R}^N \setminus \overline{D}$.

Our main result is the following

Theorem 1. The function $u$ is superharmonic in $\mathbb{R}^N$ if and only if $D$ is convex.

The “if” part of Theorem 1 must be known, at least tacitly; cf. Fuchs [1, p. 11]. For completeness we sketch a proof. For every support hyperplane $H$ of $\overline{D}$ let $u_H$ be the signed distance function from $H$ such that $u_H > 0$ in $D$ and $u_H$ is harmonic in $\mathbb{R}^N$. Then $u = \inf_H u_H$ and it follows that $u$ is superharmonic (in fact, $u$ is concave) in $\mathbb{R}^N$, since the $u_H$ are all harmonic and $u \in C(\mathbb{R}^N)$. The proof of the “only if” part of Theorem 1 (cf. §3) is more involved and requires two preliminary lemmas (§2).

We note that, for example, if $D$ is the punctured ball $D = \{X \in \mathbb{R}^N: 0 < r = ||X|| < 1\}$, then $u$ is superharmonic in $D^\prime$ but not in $D$. With this motivation we now state

Theorem 2. If $D$ is a planar domain and $d$ is superharmonic in $D$, then $D$ is convex.

In higher dimensions, neither $D$ nor $\overline{D}$ need be convex.

Theorem 3. Let $F$ be a proper closed subset of $\mathbb{R}^N$, where $N \geq 2$, and let $d$ be the distance from $\partial F$. Then $d$ is subharmonic in $F^\prime$ if and only if $F$ is convex.

The $N = 2$ and $N \geq 3$ cases of Theorem 2 are proved in §§4 and 5, Theorem 3 in §3.
2. **Lemma 1.** Let \( D \subset \mathbb{R}^N \) be such that \( D \neq \text{int}(\overline{D}) \). Then \( u \) is not superharmonic in \( \mathbb{R}^N \).

We denote the mean-value of \( u \) on \( S(\mathbf{X}_0, r) = \{ \mathbf{X} : ||\mathbf{X} - \mathbf{X}_0|| = r \} \) by \( M(u, \mathbf{X}_0, r) \).

To prove Lemma 1, choose \( \mathbf{X}_0 \in \partial D \) and \( r_0 > 0 \) so that \( B(\mathbf{X}_0, r_0) = \{ \mathbf{X} : ||\mathbf{X} - \mathbf{X}_0|| < r_0 \} \subset \text{int}(\overline{D}) \subset \overline{D} \). Clearly \( M(u, \mathbf{X}_0, r) > 0 \) if \( 0 < r < r_0 \); thus if \( u \) were superharmonic we must have \( u(\mathbf{X}_0) > 0 \).

**Lemma 2.** Let \( Y_1, Y_2 \) be distinct points in \( \mathbb{R}^N \) such that \( ||Y_1|| = ||Y_2|| \). Let \( r_1, r_2 \) denote the distances of a point from \( Y_1, Y_2 \), respectively, and define \( v \) in \( \mathbb{R}^N \) by \( v = r_1 \wedge r_2 \). Then there exists a positive number \( r_0 \) such that \( v(O) > M(v, O, r) \) for all \( r \) in \((0, r_0)\).

By using a magnification, we may suppose that \( ||Y_1|| = ||Y_2|| = 1 \), and by rotating the axes, we may suppose further that \( Y_1 = (\cos \phi, \sin \phi, 0, \ldots, 0) \) and \( Y_2 = (-\cos \phi, \sin \phi, 0, \ldots, 0) \), where \( 0 \leq \phi < \pi/2 \).

If \( \mathbf{X} = (x_1, \ldots, x_N) \in \mathbb{R}^N \) and \( r = ||\mathbf{X}|| \), then, writing

\[
/(\mathbf{X}) = r^2 - 2|x_1| \cos \phi - 2x_2 \sin \phi,
\]

we have

\[
v(\mathbf{X}) = (1 + f(\mathbf{X}))^{1/2} \leq 1 + \frac{1}{2}f(\mathbf{X}).
\]

Hence

\[
M(v, O, r) \leq 1 + \frac{1}{2}M(f, O, r) = 1 + \frac{1}{2}r^2 - (\cos \phi)M(|x_1|, O, r).
\]

Since \( M(|x_1|, O, r) \) is a positive multiple of \( r \) and \( \cos \phi > 0 \), we have \( M(v, O, r) < 1 = v(O) \) when \( r \) is small.

3. To prove the “only if” in Theorem 1, suppose that \( D \) is not convex. If \( \overline{D} \) is convex, then Lemma 1 implies that \( u \) is not superharmonic in \( \mathbb{R}^N \), since then \( \text{int}(\overline{D}) \) is convex \([2, \text{Theorem 1.11}]\) and so \( D \neq \text{int}(\overline{D}) \).

Now suppose that \( \overline{D} \) is nonconvex. A key result for this case is Motzkin’s theorem, which states that a proper closed subset \( F \) of \( \mathbb{R}^N \) is convex if and only if each point of \( \mathbb{R}^N \) has a unique nearest point of \( F \) (cf. \([2, \text{Theorem 7.8}]\)). Hence, taking \( F = \overline{D} \), we may assume that (by translating the origin, if necessary) \( O \in D' \) and that there exist distinct points \( Y_1, Y_2 \) of \( D \) such that \( d(O) = ||Y_1|| = ||Y_2|| > 0 \). Define \( v \) in \( \mathbb{R}^N \) by \( v(\mathbf{X}) = ||\mathbf{X} - Y_1|| \wedge ||\mathbf{X} - Y_2|| \). By Lemma 2, there exists \( r_0 > 0 \) such that \( v(O) > M(v, O, r) \) whenever \( 0 < r < r_0 \). Also, \( B(O, r) \subset D' \) for one of these \( r \).

Since \( v(\mathbf{X}) \geq d(\mathbf{X}) \) for all \( \mathbf{X} \in D \) with equality when \( \mathbf{X} = O \), we obtain

\[
u(O) = -d(O) = -v(O) < -M(v, O, r) \leq -M(d, O, r) = M(u, O, r),
\]

so that \( u \) is not superharmonic in \( D' \).

The argument in the last paragraph (with \( \overline{D} \) replaced by \( F \)) proves the “only if” in Theorem 3. The proof of “if” in Theorem 3 is similar to the proof of “if” in Theorem 1 (§1).

4. To prove the plane case \((N = 2)\) of Theorem 2, we suppose that \( D \) is nonconvex in \( \mathbb{R}^2 \) and show that \( d \) is not superharmonic in \( D \). There exist a point \( Y_0 \) of \( \partial D \), a positive number \( \epsilon \) and a closed half-plane \( P \) with \( Y_0 \) on \( \partial P \) such that

\[
P \cap \left( B(Y_0, \epsilon) \setminus \{Y_0\} \right) \subset D;
\]
cf. [2, Theorem 4.8]. Without loss of generality, suppose that \( Y_0 = O \) and \( P = \{X: x_2 > 0\} \). Let \( X_0 = (0, \varepsilon/4) \) and \( B = B(X_0, \varepsilon/8) \). If \( X = (x_1, x_2) \in B \) and \( x_1 \neq 0 \), then \( d(X) > x_2 \). Hence, by the mean-value equality for the function \( x_2 \),

\[
\int_B d(X) \, dX > \int_B x_2 \, dX = \pi (\varepsilon/8)^2 (\varepsilon/4) = \pi (\varepsilon/8)^2 d(X_0),
\]

so that the mean-value inequality for the superharmonicity of \( d \) fails at \( X_0 \).

5. Here we show by an example that in higher dimensions (\( N \geq 3 \)) the superharmonicity of \( u \) in \( D \) does not necessarily imply the convexity of \( D \), nor even of \( \overline{D} \).

Let \( \Omega \) denote the torus in \( \mathbb{R}^3 \) obtained by rotating the disc \( \omega = \{(0, x_2, x_3): (x_2 - a)^2 + x_3^2 < 1 \} \), where \( a \geq 2 \), about the \( x_3 \)-axis. In the case \( N = 3 \) let \( D = \Omega \), and in the case \( N > 4 \) let \( D = \Omega \times \mathbb{R}^{N-3} \). Clearly \( D \) is not convex, and neither is \( \overline{D} \).

We shall show, however, that \( d \) is superharmonic in \( D \).

With a point \( X \) (in \( \mathbb{R}^N \)) we associate plane polar coordinates \((r, \theta)\) such that \( x_1 = r \cos \theta \) and \( x_2 = r \sin \theta \) and we put \( \rho = \rho(X) = (x_2^2 + (r - a)^2)^{1/2} \). Then \( D = \{ X: \rho < 1 \} \) and \( \partial D = \{ X: \rho = 1 \} \).

If \( X \in D \), then, in finding \( d(X) \), we may suppose that \((x_1, x_2, x_3) \in \omega \). Let \( X_0 = (0, a, 0, \ldots, 0) \). Then \( B(X, 1 - \|X - X_0\|) \subset B(X_0, 1) \subset D \) and so \( d(X) > 1 - \|X - X_0\| = 1 - \rho \). If \( X = X_0 \), then clearly \( d(X) = 1 \); if \( X \neq X_0 \), then the point \( Y_0 \) such that \( \|Y_0 - X_0\| = 1 \) and \( X_0, X, Y_0 \) are collinear (in that order) belongs to \( \partial D \), so that \( d(X) \leq \|X - Y_0\| = 1 - \rho \). Hence, in all cases, \( d(X) = 1 - \rho \).

Let \( G = \{ X: \rho = 0 \} \). We show first that \( d \) is superharmonic in \( D \setminus G \) by computing the Laplacian

\[
\Delta d(X) = -\Delta \rho = -\left( \frac{\partial^2 \rho}{\partial x_2^2} + r^{-1} \frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial r^2} \right) = \frac{a - 2r}{r \rho};
\]

as we have \( 2r > 2(a - 1) \geq a \), we get \( \Delta d < 0 \). Hence \( d \) is superharmonic in \( D \setminus G \) and therefore satisfies the weak mean-value inequality in \( D \setminus G \) (that is, if \( S(X, r) \subset D \setminus G \), then \( d(X) \geq M(d, X, r) \)). Further, \( d \) takes its maximum value at each point of \( G \) and therefore the mean-value inequality holds on \( G \), too. As \( d \) is continuous, it follows that \( d \) is superharmonic in \( D \).

**REFERENCES**


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