PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 4, April 1985

WHICH AMALGAMS ARE CONVOLUTION ALGEBRAS?

JAMES STEWART[1] AND SALEEM WATSON

ABSTRACT. We determine necessary and sufficient conditions on a locally compact abelian group G for the amalgam $(L^p, l^q)(G)$ to be an algebra under convolution. If $q > 1$, G must be compact; if $p < 1$, G must be discrete. If $p \geqslant 1$ and $q \leqslant 1$, the amalgam is always an algebra.

1. Introduction. The amalgam of L^p and l^q is the space $(L^p, l^q)(G)$ consisting of all functions on a locally compact abelian group G which are locally in L^p and have l^q behavior at infinity in the sense that the L^p-norms over certain compact subsets of G form an l^q-sequence. (Precise definitions will be given in the next section.) We pose and answer the question: "Which of the amalgam spaces $(L^p, l^q)(G)$ are algebras under convolution?".

If $p = q$, then the amalgam $(L^p, l^q)(G)$ reduces to $L^p(G)$. For this case it is well known that $L^1(G)$ is always an algebra, and for $p > 1$ Żelazko **[11]** has proved that $L^p(G)$ is an algebra if and only if G is compact. For the amalgams $(L^p, l^q)(G)$ with $p \geqslant 1$, $q \geqslant 1$, it is known that $(L^p, l^1)(G)$ is always an algebra, and we prove here that $(L^p, l^q)(G)$, $q > 1$, is an algebra if and only if G is compact.

For indices smaller than 1, Żelazko **[12]** has shown that $L^p(G)$, $0 < p < 1$, is an algebra if and only if G is discrete. Likewise, for amalgams, it turns out that if $(L^p, l^q)(G)$ is an algebra for $0 < p < 1$, then G is discrete.

However, we also show that if $p \geqslant 1$ and $0 < q \leqslant 1$, then $(L^p, l^q)(G)$ is always an algebra. This provides a large class of new convolution algebras which, for $q < 1$, are F-algebras.

2. Amalgams. For functions of a real variable, Holland **[4]** defined the amalgam of L^p and l^q as the space (L^p, l^q) of functions f such that

$$\|f\|_{p,q} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right]^{1/q} < \infty,$$

although certain special cases had been studied earlier starting with Wiener **[10]**. For functions on a locally compact abelian group G, amalgams have been defined and studied by Bertrandias, Datry and Dupuis **[1]**, Stewart **[8]**, and Busby and Smith **[2]**. In particular we follow the approach of **[8]** by using the structure theorem to write

Received by the editors March 6, 1984. Presented January 10, 1985 at the annual meeting of the American Mathematical Society, Anaheim, California.

1980 *Mathematics Subject Classification*. Primary 43A15, 43A20.

Key words and phrases. Amalgam, convolution algebra.

[1] This work was supported by NSERC grant A4833.

$G = R^a \times G_1$, where a is a nonnegative integer and G_1 is a group which contains a compact open subgroup H. The Haar measure m on G_1 is normalized so that $m(H) = 1$. Define $I = [0,1)^a \times H$ and $I_\alpha = g_\alpha + I$, where each g_α is of the form $(n_1,\ldots,n_a, t)$ with $n_i \in Z$ and the t's being a transversal of H in G_1, that is, $G_1 = \bigcup_t (t + H)$. We can then write G as a disjoint union:

$$G = \bigcup_{\alpha \in J} I_\alpha.$$

In terms of this decomposition we define the amalgam $(L^p, l^q)(G)$ to be the space of functions f which are locally in L^p and are such that

$$\|f\|_{p,q} = \left[\sum_{\alpha \in J}\left[\int_{I_\alpha} |f(x)|^p\,dx\right]^{q/p}\right]^{1/q} < \infty.$$

For $p = \infty$ we have

$$\|f\|_{\infty,q} = \left[\sum_{\alpha \in J}\sup_{x \in I_\alpha} |f(x)|^q\right]^{1/q} < \infty \qquad (q < \infty).$$

We shall often use the notation f_α to mean the function which agrees with f on I_α and is 0 elsewhere. Then we can write

$$\|f\|_{p,q} = \left[\sum_{\alpha \in J} \|f_\alpha\|_p^q\right]^{1/q}.$$

Notice that if G is compact, then $(L^p, l^q)(G) = L^p(G)$. If G is discrete, then we can take $I = \{0\}$ and so $(L^p, l^q)(G) = l^q(G)$.

Previously, amalgams have been studied for $p \geqslant 1$, $q \geqslant 1$, and in this case it is known that $\|f\|_{p,q}$ is a norm and $(L^p, l^q)(G)$ is a Banach space with dual $(L^{p'}, l^{q'})(G)$, where $1/p + 1/p' = 1$, $1 \leqslant p, q < \infty$ [**1**]. However, if either p or q is less than 1, then (L^p, l^q) is an F-space. To show this we establish that $\|f\|_{p,q}$ is a quasinorm by using the following inequalities [**5**, p. 158]:

(2.1) $$(a+b)^p \leqslant 2^{p-1}(a^p + b^p) \qquad (p > 1),$$

(2.2) $$(a+b)^p \leqslant a^p + b^p \qquad (0 < p < 1),$$

(2.3) $$\|f+g\|_p \leqslant 2^{(1-p)/p}\left(\|f\|_p + \|g\|_p\right) \qquad (0 < p < 1).$$

For $p \geqslant 1$, $q < 1$, we have

$$\begin{aligned}\|f+g\|_{p,q}^q &= \sum_{\alpha \in J} \|(f+g)_\alpha\|_p^q \leqslant \sum_{\alpha \in J}\left(\|f_\alpha\|_p + \|g_\alpha\|_p\right)^q \\ &\leqslant \sum_{\alpha \in J}\|f_\alpha\|_p^q + \sum_{\alpha \in J}\|g_\alpha\|_p^q \quad \text{[by (2.2)]} \\ &= \|f\|_{p,q}^q + \|g\|_{p,q}^q.\end{aligned}$$

Then (2.1) gives

(2.4) $$\|f+g\|_{p,q} \leqslant 2^{(1-q)/q}\left(\|f\|_{p,q} + \|g\|_{p,q}\right) \qquad (p \geqslant 1, q < 1).$$

Similarly, using (2.1), (2.2) and (2.3) we obtain

$$\|f+g\|_{p,q} \leqslant 2^{(1-p)/p}\left(\|f\|_{p,q}+\|g\|_{p,q}\right) \qquad (p<1,\, q \geqslant 1), \tag{2.5}$$

$$\|f+g\|_{p,q} \leqslant 2^{(1-p)/p}2^{(1-q)/q}\left(\|f\|_{p,q}+\|g\|_{p,q}\right) \qquad (p<1,\, q<1). \tag{2.6}$$

The inequalities (2.4), (2.5) and (2.6) show that when either p or q is less than 1, $(L^p, l^q)(G)$ is a quasi-normed space which is therefore locally bounded [**5**, p. 159] and hence an F-space [**7**].

The following relations were given in [**8**] for $p, q \geqslant 1$ but continue to hold when either index is less than 1:

$$(L^p, l^{q_1}) \subset (L^p, l^{q_2}) \qquad (0<p\leqslant\infty,\, 0<q_1\leqslant q_2\leqslant\infty), \tag{2.7}$$

$$(L^{p_2}, l^{q}) \subset (L^{p_1}, l^{q}) \qquad (0<p_1\leqslant p_2\leqslant\infty,\, 0<q\leqslant\infty), \tag{2.8}$$

$$(L^p, l^p) = L^p \qquad (0<p\leqslant\infty), \tag{2.9}$$

$$(L^p, l^q) \subset L^p \cap L^q \qquad (0<q\leqslant p\leqslant\infty), \tag{2.10}$$

$$L^p \cup L^q \subset (L^p, l^q) \qquad (0<p\leqslant q\leqslant\infty). \tag{2.11}$$

Associated with (2.7) and (2.8) are the inequalities

$$\|f\|_{p,q_2} \leqslant \|f\|_{p,q_1} \qquad (0<p\leqslant\infty,\, 0<q_1\leqslant q_2\leqslant\infty), \tag{2.12}$$

$$\|f\|_{p_1,q} \leqslant \|f\|_{p_2,q} \qquad (0<p_1\leqslant p_2\leqslant\infty,\, 0<q\leqslant\infty). \tag{2.13}$$

3. The case $p \geqslant 1$, $q \geqslant 1$. If we combine Young's Inequality for amalgams [**1** or **2**] with the inequality (2.13), we get

$$\|f * g\|_{p,1} \leqslant C\|f\|_1\|g\|_{p,1} \leqslant C\|f\|_{p,1}\|g\|_{p,1},$$

and this shows that $(L^p, l^1)(G)$ is a Banach algebra for $p \geqslant 1$ and for any G. (In fact, it is a Segal algebra [**6**, **1**]. The classic special case is the Wiener algebra $(L^\infty, l^1) \cap C(R)$ as studied by Goldberg [**3**].)

We now consider the case $p \geqslant 1$, $q > 1$. Here $(L^p, l^q)(G)$ is a Banach space, so if it is a topological algebra, then it is well known that there is an equivalent submultiplicative norm $\|\cdot\|$ such that $\|f\| \geqslant \|f\|_{p,q}$. The proof of the following theorem is adapted from Urbanik [**9**].

THEOREM 1. *The amalgam* $(L^p, l^q)(G)$, $p \geqslant 1$, $q > 1$, *is a topological algebra if and only if G is compact.*

PROOF. If G is compact, then $(L^p, l^q)(G) = L^p(G)$ which is known to be an algebra [**11**].

Now suppose that $A = (L^p, l^q)(G)$ is a topological algebra. We first show that $A \neq \operatorname{rad}(A)$. Let V be a symmetric, compact neighborhood of 0 in G and let χ be the characteristic function of $V+V$. Then, for $x \in V$, we have

$$\chi^{n+1}(x) = \int_G \chi(x-y)\,\chi^n(y)\,dy \geqslant \int_V \chi^n(y)\,dy,$$

where χ^n means the n-fold convolution of χ. By induction it follows that $\chi^{n+1}(x) \geqslant [m(V)]^n$ when $x \in V$. Putting $c = 1/\|\chi\|$, we obtain

$$\begin{aligned}
\|\chi^n\| &\geqslant c\|\chi^{n+1}\| \geqslant c\|\chi^{n+1}\|_{p,q} \\
&= c\left[\sum_{\alpha\in J}\left[\int_{I_\alpha} |\chi^{n+1}|^p\right]^{q/p}\right]^{1/q} \geqslant c\left[\sum_{\alpha}\left[\int_{I_\alpha\cap V} |\chi^{n+1}|^p\right]^{q/p}\right]^{1/q} \\
&\geqslant c\left[\sum_{\alpha}\left[\int_{I_\alpha\cap V} [m(V)]^{np}\right]^{q/p}\right]^{1/q} = c\left[\sum_{\alpha} m(V)^{nq} m(V\cap I_\alpha)^{q/p}\right]^{1/q} \\
&\geqslant c[m(V)]^n\left[\sum_{\alpha}[m(V\cap I_\alpha)]^{q/p}\right]^{1/q} > 0.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|\chi^n\|^{1/n} \geqslant m(V) > 0,$$

and so $\chi \notin \operatorname{Rad}(A)$. Therefore there is a nontrivial multiplicative linear functional T on A. Since T is continuous there exists a function f in $(L^{p'}, l^{q'})$ with $\|f\|_{p',q'} > 0$ such that

$$T(\phi) = \int_G f(x)\phi(x)\,dx \qquad (\phi \in (L^p, l^q)).$$

Since T is multiplicative, for every $\phi, \psi \in (L^p, l^q)$ we have

$$\begin{aligned}
\int_G\int_G f(x+y)\phi(x)\psi(y)\,dx\,dy &= \int_G f(x)\left[\int_G \phi(x-y)\psi(y)\,dy\right]dx \\
&= \int_G f(x)(\phi*\psi)(x)\,dx = T(\phi*\psi) = T(\phi)T(\psi) \\
&= \int_G\int_G f(x)f(y)\phi(x)\psi(y)\,dx\,dy.
\end{aligned}$$

This shows that

$$f(x+y) = f(x)f(y) \quad (l.\text{ a.e.}).$$

Now we use the fact that (L^p, l^q) has an equivalent translation-invariant norm $\|\cdot\|^{\#}_{p,q}$ [**1**, Proposition VIII]:

$$c''\|f\|^{\#}_{p,q} \leqslant \|f\|_{p,q} \leqslant c'\|f\|p^{\#}_{p,q}.$$

It follows that translation is a bounded operator on (L^p, l^q):

$$\|f_y\|_{p,q} \leqslant K\|f\|_{p,q},$$

where $f_y(x) = f(x+y)$. Applying this to $(L^{p'}, l^{q'})$, we have

$$\begin{aligned}
\|f\|_{p',q'} \leqslant K\|f_y\|_{p',q'} &= K\left[\sum_{\alpha\in J}\left[\int_{I_\alpha} |f(x+y)|^{p'}\,dx\right]^{q'/p'}\right]^{1/q'} \\
&= K|f(y)|\left[\sum_{\alpha\in J}\left[\int_{I_\alpha} |f(x)|^{p'}\,dx\right]^{q'/p'}\right]^{1/q'} = K|f(y)|\,\|f\|_{p',q'}.
\end{aligned}$$

Since $\|f\|_{p',q'} \neq 0$, we conclude that $|f(y)| \geqslant 1/K$ on G. This shows that the constant functions belong to $(L^{p'}, l^{q'})$. Since $q > 1$, we have $q' \neq \infty$, and so we can write

$$\infty > \|1\|_{p',q'}^{q'} = \sum_{\alpha \in J} [m(I_\alpha)]^{q'/p'} = \sum_{\alpha \in J} m(I_\alpha) = m(G).$$

Thus G is compact.

4. The case $p < 1$. For any nondiscrete group G, Żelazko **[12]** has constructed two functions f and g in $L^p(G)$ whose convolution $f * g$ is infinite on a set of positive measure. These functions f and g have compact support and so they belong to $(L^p, l^q)(G)$ for any $q > 0$. Therefore we have the following theorem.

THEOREM 2. *If the amalgam* $(L^p, l^q)(G)$, $0 < p < 1$, *is a topological algebra, then* G *is discrete.*

In order to give a converse, we first note that if G is discrete, then $(L^p, l^q)(G) = l^q(G)$. If $q \leqslant 1$, then $l^q(G)$ is a topological algebra **[12]**. If $q > 1$ and $l^q(G)$ is an algebra, then G is both discrete and compact, hence finite. And if G is finite, $l^q(G)$ is always an algebra. We have thus proved the following corollaries of Theorem 2.

COROLLARY 1. *The amalgam* $(L^p, l^q)(G)$, $0 < p < 1$, $0 < q \leqslant 1$, *is a topological algebra if and only if* G *is discrete.*

COROLLARY 2. *The amalgam* $(L^p, l^q)(G)$, $0 < p < 1$, $q > 1$, *is a topological algebra if and only if* G *is finite.*

5. The case $p \geqslant 1$, $q \leqslant 1$. We have seen that for $(L^p, l^q)(G)$ to be a topological algebra, G must be compact if $q > 1$ and G must be discrete if $p < 1$. By contrast with this situation, we show that in the remaining case ($p \geqslant 1$, $q \leqslant 1$) $(L^p, l^q)(G)$ is always an algebra.

THEOREM 3. *The amalgam* $(L^p, l^q)(G)$, $p \geqslant 1$, $0 < q \leqslant 1$, *is a topological algebra under convolution for any locally compact abelian group* G. *Moreover, for* f *and* g *in* $(L^p, l^q)(G)$ *we have*

$$\|f * g\|_{p,q} \leqslant 2^{a/q} \|f\|_{p,q} \|g\|_{p,q},$$

where a *is given by the structure theorem as in* §2.

PROOF. We first observe that, in the notation introduced in §2,

$$(f * g)(x) = \sum_{\alpha \in J} \sum_{\beta \in J} (f_\alpha * g_\beta)(x).$$

Therefore, for any $\gamma \in J$, Minkowski's Inequality gives

$$\|(f * g)_\gamma\|_p = \left\| \sum_\alpha \sum_\beta (f_\alpha * g_\beta)_\gamma \right\|_p \leqslant \sum_\alpha \sum_\beta \|(f_\alpha * g_\beta)_\gamma\|_p$$
$$\leqslant \sum_{\substack{\alpha\beta \\ \gamma \subset \alpha + \beta}} \|(f_\alpha * g_\beta)\|_p,$$

where we use the notation $\gamma \subset \alpha + \beta$ to mean that we sum over all indices α and β in J such that $I_\gamma \subset I_\alpha + I_\beta$. Now we apply Young's Inequality for L^p-spaces to write

$$\|(f * g)_\gamma\|_p \leqslant \sum_{\substack{\alpha\,\beta \\ \gamma \subset \alpha+\beta}}\sum \|f_\alpha\|_1 \|g_\beta\|_p.$$

Using the fact that $q \leqslant 1$ together with (2.2), we have

$$\|(f * g)_\gamma\|_p^q \leqslant \left(\sum_{\substack{\alpha\,\beta \\ \gamma \subset \alpha+\beta}}\sum \|f_\alpha\|_1 \|g_\beta\|_p \right)^q \leqslant \sum_{\substack{\alpha\,\beta \\ \gamma \subset \alpha+\beta}}\sum \|f_\alpha\|_1^q \|g_\beta\|_p^q,$$

and so

$$\begin{aligned} \|f * g\|_{p,q}^q &= \sum_\gamma \|(f * g)_\gamma\|_p^q < \sum_\gamma \sum_\alpha \sum_{\substack{\beta \\ \gamma \subset \alpha+\beta}} \|f_\alpha\|_1^q \|g_\beta\|_p^q \\ &= \sum_\alpha \|f_\alpha\|_1^q \sum_\gamma \sum_{\substack{\beta \\ \gamma \subset \alpha+\beta}} \|g_\beta\|_p^q \leqslant 2^a \|f\|_{1,q}^q \|g\|_{p,q}^q. \end{aligned}$$

Note the presence of the factor 2^a. This is because, for fixed α and γ, there are 2^a I_β's such that $I_\gamma \subset I_\alpha + I_\beta$, and as we sum over γ each such β occurs exactly 2^a times. Finally, taking qth roots, we obtain

$$\begin{aligned} \|f * g\|_{p,q} &\leqslant 2^{a/q} \|f\|_{1,q} \|g\|_{p,q} \\ &\leqslant 2^{a/q} \|f\|_{p,q} \|g\|_{p,q} \quad [\text{by } (2.13)]. \end{aligned}$$

This completes the proof.

In view of the fact, established in §2, that $(L^p, l^q)(G)$ is an F-space whenever either index is less than 1, Theorem 3 says that $(L^p, l^q)(G)$, $p \geqslant 1$, $q \leqslant 1$, is an F-algebra.

Notice from the proof of Theorem 3 that

$$\|f * g\|_{p,q} \leqslant 2^{a/q} \|f\|_{1,q} \|g\|_{p,q},$$

and this shows that $(L^p, l^q)(G)$ is an ideal in the algebra $(L^1, l^q)(G)$.

REFERENCES

1. J.-P. Bertrandias, C. Datry and C. Dupuis, *Unions et intersections d'èspaces L^p invariantes par translation où convolution*, Ann. Inst. Fourier (Grenoble) **28** (1978), 53–84.n

2. R. C. Busby and H. A. Smith, *Product-convolution operators and mixed-norm spaces*, Trans. Amer. Math. Soc. **263** (1981), 309–341.

3. R. R. Goldberg, *On a space of functions of Wiener*, Duke Math. J. **34** (1967), 683–692.

4. F. Holland, *Harmonic analysis on amalgams of L^p and l^q*, J. London Math. Soc. (2) **10** (1975), 295–305.

5. G. Köthe, *Topological vector spaces*, Springer-Verlag, New York, 1969.

6. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Univ. Press, New York and London, 1968.

7. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

8. J. Stewart, *Fourier transforms of unbounded measures*, Canad. J. Math. **31** (1979), 1281–1292.

9. K. Urbanik, *A proof of a theorem of Żelazko on L^p-algebras*, Colloq. Math. **8** (1961), 121–123.

10. N. Wiener, *On the representation of functions by trigonometric integrals*, Math. Z. **24** (1926), 575–616.
11. W. Żelazko, *On the algebras L_p of locally compact groups*, Colloq. Math. **8** (1961), 115–120.
12. ______, *A theorem on the discrete groups and algebras L_p*, Colloq. Math **8** (1961), 205–207.

DEPARTMENT OF MATHEMATICAL SCIENCES, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, L8S 4K1 CANADA

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802