WHICH AMALGAMS ARE CONVOLUTION ALGEBRAS?

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Abstract. We determine necessary and sufficient conditions on a locally compact abelian group $G$ for the amalgam $(L^p, l^q)(G)$ to be an algebra under convolution. If $q > 1$, $G$ must be compact; if $p < 1$, $G$ must be discrete. If $p \geq 1$ and $q \leq 1$, the amalgam is always an algebra.

1. Introduction. The amalgam of $L^p$ and $l^q$ is the space $(L^p, l^q)(G)$ consisting of all functions on a locally compact abelian group $G$ which are locally in $L^p$ and have $l^q$ behavior at infinity in the sense that the $L^p$-norms over certain compact subsets of $G$ form an $l^q$-sequence. (Precise definitions will be given in the next section.) We pose and answer the question: "Which of the amalgam spaces $(L^p, l^q)(G)$ are algebras under convolution?".

If $p = q$, then the amalgam $(L^p, l^q)(G)$ reduces to $L^p(G)$. For this case it is well known that $L^1(G)$ is always an algebra, and for $p > 1$ Żelazko [11] has proved that $L^p(G)$ is an algebra if and only if $G$ is compact. For the amalgams $(L^p, l^q)(G)$ with $p \geq 1$, $q \geq 1$, it is known that $(L^p, l^1)(G)$ is always an algebra, and we prove here that $(L^p, l^q)(G)$, $q > 1$, is an algebra if and only if $G$ is compact.

For indices smaller than 1, Żelazko [12] has shown that $L^p(G)$, $0 < p < 1$, is an algebra if and only if $G$ is discrete. Likewise, for amalgams, it turns out that if $(L^p, l^q)(G)$ is an algebra for $0 < p < 1$, then $G$ is discrete.

However, we also show that if $p \geq 1$ and $0 < q \leq 1$, then $(L^p, l^q)(G)$ is always an algebra. This provides a large class of new convolution algebras which, for $q < 1$, are $F$-algebras.

2. Amalgams. For functions of a real variable, Holland [4] defined the amalgam of $L^p$ and $l^q$ as the space $(L^p, l^q)$ of functions $f$ such that

$$
\|f\|_{L^p, l^q} = \left[ \sum_{n=-\infty}^{\infty} \left( \int_n^{n+1} |f(x)|^q \, dx \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty,
$$

although certain special cases had been studied earlier starting with Wiener [10]. For functions on a locally compact abelian group $G$, amalgams have been defined and studied by Bertrandias, Datry and Dupuis [1], Stewart [8], and Busby and Smith [2]. In particular we follow the approach of [8] by using the structure theorem to write
$G = R^a \times G_1$, where $a$ is a nonnegative integer and $G_1$ is a group which contains a compact open subgroup $H$. The Haar measure $m$ on $G_1$ is normalized so that $m(H) = 1$. Define $I = \{0, 1\}^a \times H$ and $I_a = g_a + I$, where each $g_a$ is of the form $(n_1, \ldots, n_q, t)$ with $n_i \in Z$ and the $t$'s being a transversal of $H$ in $G_1$, that is, $G_1 = \bigcup_1^q (t + H)$. We can then write $G$ as a disjoint union:

$$G = \bigcup_{\alpha \in J} I_\alpha.$$

In terms of this decomposition we define the amalgam $(L^p, l^q)(G)$ to be the space of functions $f$ which are locally in $L^p$ and are such that

$$\|f\|_{p,q} = \left[ \sum_{\alpha \in J} \left( \int_{I_\alpha} |f(x)|^p \, dx \right)^{q/p} \right]^{1/q} < \infty.$$

For $p = \infty$ we have

$$\|f\|_{\infty,q} = \left[ \sum_{\alpha \in J} \sup_{x \in I_\alpha} |f(x)|^q \right]^{1/q} < \infty \quad (q < \infty).$$

We shall often use the notation $f_\alpha$ to mean the function which agrees with $f$ on $I_\alpha$ and is 0 elsewhere. Then we can write

$$\|f\|_{p,q} = \left[ \sum_{\alpha \in J} \|f_\alpha\|_p^q \right]^{1/q}.$$

Notice that if $G$ is compact, then $(L^p, l^q)(G) = L^p(G)$. If $G$ is discrete, then we can take $I = \{0\}$ and so $(L^p, l^q)(G) = l^q(G)$.

Previously, amalgams have been studied for $p \geq 1$, $q \geq 1$, and in this case it is known that $\|f\|_{p,q}$ is a norm and $(L^p, l^q)(G)$ is a Banach space with dual $(L^{p'}, l^{q'})$, where $1/p + 1/p' = 1$, $1 \leq p, q < \infty$ [1]. However, if either $p$ or $q$ is less than 1, then $(L^p, l^q)$ is an $F$-space. To show this we establish that $\|f\|_{p,q}$ is a quasinorm by using the following inequalities [5, p. 158]:

\begin{align*}
(2.1) & \quad (a + b)^p \leq 2^{p-1}(a^p + b^p) \quad (p > 1), \\
(2.2) & \quad (a + b)^p \leq a^p + b^p \quad (0 < p < 1), \\
(2.3) & \quad \|f + g\|_p \leq 2^{(1-p)/p} (\|f\|_p + \|g\|_p) \quad (0 < p < 1).
\end{align*}

For $p \geq 1$, $q < 1$, we have

\begin{align*}
\|f + g\|_{p,q}^q &= \sum_{\alpha \in J} \|(f + g)_\alpha\|_p^q \leq \sum_{\alpha \in J} \left( \|f_\alpha\|_p + \|g_\alpha\|_p \right)^q \\
&\leq \sum_{\alpha \in J} \|f_\alpha\|_p^q + \sum_{\alpha \in J} \|g_\alpha\|_p^q \quad \text{[by (2.2)]} \\
&= \|f\|_{p,q}^q + \|g\|_{p,q}^q.
\end{align*}

Then (2.1) gives

$$\|f + g\|_{p,q} \leq 2^{(1-q)/q} (\|f\|_{p,q} + \|g\|_{p,q}) \quad (p \geq 1, q < 1).$$
Similarly, using (2.1), (2.2) and (2.3) we obtain

\[(2.5) \quad \|f + g\|_{p,q} \leq 2^{1-p_1/p} (\|f\|_{p,q} + \|g\|_{p,q}) \quad (p < 1, q \geq 1),\]

\[(2.6) \quad \|f + g\|_{p,q} \leq 2^{(1-p_1/p)2^{(1-q)/q}} (\|f\|_{p,q} + \|g\|_{p,q}) \quad (p < 1, q < 1).\]

The inequalities (2.4), (2.5) and (2.6) show that when either \(p\) or \(q\) is less than 1, \((L^p, l^q)(G)\) is a quasi-normed space which is therefore locally bounded [5, p. 159] and hence an \(F\)-space [7].

The following relations were given in [8] for \(p, q \geq 1\) but continue to hold when either index is less than 1:

\[(2.7) \quad (L^p, l^{q_1}) \subseteq (L^p, l^{q_2}) \quad (0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty),\]

\[(2.8) \quad (L^{p_1}, l^q) \subseteq (L^{p_2}, l^q) \quad (0 < p_1 \leq p_2 \leq \infty, 0 < q \leq \infty),\]

\[(2.9) \quad (L^p, l^p) = L^p \quad (0 < p \leq \infty),\]

\[(2.10) \quad (L^p, l^q) \subseteq L^p \cap L^q \quad (0 < q < p \leq \infty),\]

\[(2.11) \quad L^p \cup L^q \subseteq (L^p, l^q) \quad (0 < p \leq q \leq \infty).\]

Associated with (2.7) and (2.8) are the inequalities

\[(2.12) \quad \|f\|_{p,q_2} \leq \|f\|_{p,q_1} \quad (0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty),\]

\[(2.13) \quad \|f\|_{p_1,q} \leq \|f\|_{p_2,q} \quad (0 < p_1 \leq p_2 \leq \infty, 0 < q \leq \infty).\]

**3. The case \(p \geq 1, q \geq 1\).** If we combine Young’s Inequality for amalgams [1 or 2] with the inequality (2.13), we get

\[\|f \ast g\|_{p,1} \leq C\|f\|_p \|g\|_p,\]

and this shows that \((L^p, l^1)(G)\) is a Banach algebra for \(p \geq 1\) and for any \(G\). (In fact, it is a Segal algebra [6, 1]. The classic special case is the Wiener algebra \((L^\infty, l^1) \cap C(R)\) as studied by Goldberg [3].)

We now consider the case \(p \geq 1, q > 1\). Here \((L^p, l^q)(G)\) is a Banach space, so if it is a topological algebra, then it is well known that there is an equivalent submultiplicative norm \(\| \cdot \|\) such that \(\|f\| \leq \|f\|_{p,q}\). The proof of the following theorem is adapted from Urbanik [9].

**Theorem 1.** The amalgam \((L^p, l^q)(G), p \geq 1, q > 1\), is a topological algebra if and only if \(G\) is compact.

**Proof.** If \(G\) is compact, then \((L^p, l^q)(G) = L^p(G)\) which is known to be an algebra [11].

Now suppose that \(A = (L^p, l^q)(G)\) is a topological algebra. We first show that \(A \neq \text{rad}(A)\). Let \(V\) be a symmetric, compact neighborhood of 0 in \(G\) and let \(\chi\) be the characteristic function of \(V + V\). Then, for \(x \in V\), we have

\[\chi^{n+1}(x) = \int_G \chi(x - y) \chi^n(y) dy \geq \int_V \chi^n(y) dy.\]
where $\chi^n$ means the $n$-fold convolution of $\chi$. By induction it follows that $\chi^{n+1}(x) \geq [m(V)]^n$ when $x \in V$. Putting $c = 1/||x||$, we obtain

$$
\|\chi^n\| \geq c\|\chi^{n+1}\| = c\|\chi^{n+1}\|_{p,q}
$$

$$
= c \left[ \sum_{a \in J} \left[ \int_{I_a} |\chi^{n+1}|^p \right]^{q/p} \right]^{1/q} \geq c \left[ \sum_{a \in J} \left[ \int_{I_a \cap V} |m(V)|^n \right]^{q/p} \right]^{1/q}
$$

$$
\geq c \left[ \sum_{a \in J} \left[ \int_{I_a \cap V} m(V)^n m(V \cap I_a)^{q/p} \right] \right]^{1/q}
$$

Thus

$$
\lim_{n \to \infty} \|\chi^n\|^{1/n} \geq m(V) > 0,
$$

and so $\chi \not\in \text{Rad}(A)$. Therefore there is a nontrivial multiplicative linear functional $T$ on $A$. Since $T$ is continuous there exists a function $f$ in $(L^{p''}, l^{q''})$ with $\|f\|_{p',q'} > 0$ such that

$$
T(\phi) = \int_G f(x)\phi(x) \, dx \quad (\phi \in (L^p, l^q)).
$$

Since $T$ is multiplicative, for every $\phi, \psi \in (L^p, l^q)$ we have

$$
\int_G \int_G f(x + y)\phi(x)\psi(y) \, dx \, dy = \int_G \int_G f(x) \left[ \int_G \phi(x - y)\psi(y) \, dy \right] \, dx
$$

$$
= \int_G \int_G f(x)(\phi \ast \psi)(x) \, dx = T(\phi \ast \psi) = T(\phi)T(\psi)
$$

$$
= \int_G \int_G f(x)f(y)\phi(x)\psi(y) \, dx \, dy.
$$

This shows that

$$
f(x + y) = f(x)f(y) \quad (l. \ a.e.).
$$

Now we use the fact that $(L^p, l^q)$ has an equivalent translation-invariant norm $\| \cdot \|^#_{p,q}$ [1, Proposition VIII]:

$$
c''\|f\|^#_{p,q} \leq \|f\|_{p,q} \leq c'\|f\|^#_{p,q}.
$$

It follows that translation is a bounded operator on $(L^p, l^q)$:

$$
\|f(x + y)\|_{p',q'} \leq K\|f\|_{p',q'}
$$

where $f(y) = f(x + y)$. Applying this to $(L^{p''}, l^{q''})$, we have

$$
\|f\|_{p'',q'} \leq K\|f\|_{p',q'} = K \left[ \sum_{a \in J} \left[ \int_{I_a} |f(x + y)|^{q'/p'} \, dx \right]^{q'/q''} \right]^{1/q''}
$$

$$
= K|f(y)| \left[ \sum_{a \in J} \left[ \int_{I_a} |f(x)|^{q'/p'} \, dx \right]^{q'/q''} \right]^{1/q'} = K|f(y)| \|f\|_{p',q'}.
$$
Since \( \|f\|_{p', q'} \neq 0 \), we conclude that \( |f(y)| \geq 1/K \) on \( G \). This shows that the constant functions belong to \((L^p, l^q)\). Since \( q > 1 \), we have \( q' \neq \infty \), and so we can write

\[
\infty > \|1\|_{p', q'}^{q'} = \sum_{\alpha \in J} [m(I_\alpha)]^{q'/p'} = \sum_{\alpha \in J} m(I_\alpha) = m(G).
\]

Thus \( G \) is compact.

4. The case \( p < 1 \). For any nondiscrete group \( G \), Želazko [12] has constructed two functions \( f \) and \( g \) in \( L^p(G) \) whose convolution \( f \ast g \) is infinite on a set of positive measure. These functions \( f \) and \( g \) have compact support and so they belong to \((L^p, l^q)(G)\) for any \( q > 0 \). Therefore we have the following theorem.

**Theorem 2.** If the amalgam \((L^p, l^q)(G)\), \( 0 < p < 1 \), is a topological algebra, then \( G \) is discrete.

In order to give a converse, we first note that if \( G \) is discrete, then \((L^p, l^q)(G) = l^q(G)\). If \( q \leq 1 \), then \( l^q(G) \) is a topological algebra [12]. If \( q > 1 \) and \( l^q(G) \) is an algebra, then \( G \) is both discrete and compact, hence finite. And if \( G \) is finite, \( l^q(G) \) is always an algebra. We have thus proved the following corollaries of Theorem 2.

**Corollary 1.** The amalgam \((L^p, l^q)(G)\), \( 0 < p < 1 \), \( 0 < q \leq 1 \), is a topological algebra if and only if \( G \) is discrete.

**Corollary 2.** The amalgam \((L^p, l^q)(G)\), \( 0 < p < 1 \), \( q > 1 \), is a topological algebra if and only if \( G \) is finite.

5. The case \( p \geq 1 \), \( q \leq 1 \). We have seen that for \((L^p, l^q)(G)\) to be a topological algebra, \( G \) must be compact if \( q > 1 \) and \( G \) must be discrete if \( p < 1 \). By contrast with this situation, we show that in the remaining case \((p \geq 1, q \leq 1)\) \((L^p, l^q)(G)\) is always an algebra.

**Theorem 3.** The amalgam \((L^p, l^q)(G)\), \( p \geq 1 \), \( 0 < q \leq 1 \), is a topological algebra under convolution for any locally compact abelian group \( G \). Moreover, for \( f \) and \( g \) in \((L^p, l^q)(G)\) we have

\[
\|f \ast g\|_{p, q} \leq 2^{a/q} \|f\|_{p, q} \|g\|_{p, q},
\]

where \( a \) is given by the structure theorem as in §2.

**Proof.** We first observe that, in the notation introduced in §2,

\[
(f \ast g)(x) = \sum_{\alpha \in J} \sum_{\beta \in J} (f_\alpha \ast g_\beta)(x).
\]

Therefore, for any \( \gamma \in J \), Minkowski’s Inequality gives

\[
\|(f \ast g)_{\gamma}\|_p = \sum_{\alpha} \sum_{\beta} \|(f_\alpha \ast g_\beta)_{\gamma}\|_p \leq \sum_{\alpha} \sum_{\beta} \|(f_\alpha \ast g_\beta)_{\gamma}\|_p \leq \sum_{\alpha} \sum_{\beta} \|(f_\alpha \ast g_\beta)\|_p.
\]
where we use the notation $\gamma \subseteq \alpha + \beta$ to mean that we sum over all indices $\alpha$ and $\beta$ in $J$ such that $I_{\gamma} \subset I_{\alpha} + I_{\beta}$. Now we apply Young’s Inequality for $L^p$-spaces to write

$$\| (f \ast g) \gamma \|_p \leq \sum_{\alpha, \beta} \| f_\alpha \|_1 \| g_\beta \|_p,$$

Using the fact that $q \leq 1$ together with (2.2), we have

$$\| (f \ast g) \gamma \|_p^q \leq \left( \sum_{\alpha, \beta} \| f_\alpha \|_1 \| g_\beta \|_p \right)^q \leq \sum_{\alpha, \beta} \| f_\alpha \|_1^q \| g_\beta \|_p^q,$$

and so

$$\| f \ast g \|_{p,q}^q = \sum_\gamma \| (f \ast g) \gamma \|_p^q < \sum_\gamma \sum_\alpha \sum_\beta \| f_\alpha \|_1^q \| g_\beta \|_p^q$$

$$\leq \sum_\alpha \sum_\gamma \sum_\beta \| f_\alpha \|_1^q \| g_\beta \|_p^q \leq 2^a \| f \|_{1,q} \| g \|_{p,q}.$$  

Note the presence of the factor $2^a$. This is because, for fixed $\alpha$ and $\gamma$, there are $2^a I_\beta$’s such that $I_{\gamma} \subset I_{\alpha} + I_{\beta}$, and as we sum over $\gamma$ each such $\beta$ occurs exactly $2^a$ times. Finally, taking $q$th roots, we obtain

$$\| f \ast g \|_{p,q} \leq 2^{a/q} \| f \|_{1,q} \| g \|_{p,q} \leq 2^{a/q} \| f \|_{p,q} \| g \|_{p,q} \quad \text{[by (2.13)].}$$

This completes the proof.

In view of the fact, established in §2, that $(L^p, l^q)(G)$ is an $F$-space whenever either index is less than 1, Theorem 3 says that $(L^p, l^q)(G)$, $p \geq 1$, $q \leq 1$, is an $F$-algebra.

Notice from the proof of Theorem 3 that

$$\| f \ast g \|_{p,q} \leq 2^{a/q} \| f \|_{1,q} \| g \|_{p,q},$$

and this shows that $(L^p, l^q)(G)$ is an ideal in the algebra $(L^1, l^q)(G)$.

REFERENCES


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