ON FIXED POINTS OF LINEAR CONTRACTIONS

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Abstract. It is shown that a weakly closed convex semigroup of linear contractions on a separable Hilbert space has a common fixed point other than 0 if the operator 0 is not in the semigroup.

We prove a theorem on existence of common fixed points for certain convex semigroups of linear operators on Banach spaces. The special case where the semigroup is a group follows easily from Kakutani's well-known theorem [4, 5, 6] and also, as discussions with P. Milman revealed, from the work of Brodskii and Milman [1]. Similarly, in the case where the semigroup is commutative, our result is a corollary of a special case of the Markov-Kakutani theorem [3, 4]. Nonetheless, it appears that the results and corollaries given below have not been noticed before. Corollary 4, for example, gives a sufficient condition that $V_{n=N}^\infty \{A^n\}$ be the same for all $N$.

The applications of the fixed-point theorem that we consider concern operators on Hilbert space, but it seems worthwhile to state the theorem more generally.

Theorem 1. Let $\mathcal{X}$ be a strictly convex reflexive Banach space, and let $\mathcal{S}$ be a weak operator closed separable convex semigroup of linear contractions on $\mathcal{X}$. Then the operators in $\mathcal{S}$ have a common fixed point other than 0 if and only if the operator 0 is not in $\mathcal{S}$.

Proof. Clearly, if the operator 0 is in $\mathcal{S}$, then the only common fixed point is 0.

To prove the converse first recall that $(T_n) \to T$ in the weak operator topology if and only if $\phi(T_n x) \to \phi(T x)$, for each $\phi \in \mathcal{X}^*$ and $x \in \mathcal{X}$. We require the fact that the unit ball of $\mathcal{B}(\mathcal{X})$ is weak operator compact; this can be proven as in the better-known case of Hilbert space. (That is, consider the Cartesian product of the closed balls of radius $\|x\|$ in $\mathcal{X}$, indexed by $\mathcal{X}$, where each ball is given the weak topology).

Let $\{T_n\}_{n=1}^\infty$ be a countable weak operator dense subset of $\mathcal{S}$; it obviously suffices to find a common fixed point for the $\{T_n\}$. Let

$$T = \sum_{n=1}^\infty \frac{1}{2^n} T_n;$$
this series converges in the norm topology (hence also in the weak operator
topology) of \( B(\mathcal{X}) \), and the closed convexity of \( \mathcal{S} \) implies \( T \in \mathcal{S} \). Now \( T \) defines a
mapping of \( \mathcal{S} \) into itself by \( T(S) = TS \) for \( S \in \mathcal{S} \) (\( \mathcal{S} \) is a semigroup). Since \( \mathcal{S} \) is a
compact convex set, Schauder’s fixed point theorem yields an operator \( S_0 \in \mathcal{S} \) such
that \( TS_0 = S_0 \). Choose \( x \in \mathcal{X} \) such that \( S_0x \neq 0 \). Then
\[
\sum_{n=1}^{\infty} \frac{1}{2^n} T_n S_0 x = S_0 x.
\]
For each \( n_0 \),
\[
\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x + \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| = \|S_0 x\|,
\]
\[
\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x \right\| \leq \left( \sum_{n \neq n_0} \frac{1}{2^n} \right) \|S_0 x\|,
\]
and
\[
\left\| \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| \leq \frac{1}{2^{n_0}} \|S_0 x\|
\]
imply that the above inequalities are equations, so the strict convexity of \( \mathcal{X} \) implies
that \( T_{n_0} S_0 x \) is a multiple of \( \sum_{n \neq n_0} T_n S_0 x / 2^n \). Hence, \( T_{n_0} S_0 x \) is a multiple of \( S_0 x \).
(Recall that \( \mathcal{X} \) strictly convex means that \( \|x_1 + x_2\| = \|x_1\| + \|x_2\| \) implies \( \{x_1, x_2\} \)
is linearly dependent). Thus, \( T_n S_0 x \) is a multiple of \( S_0 x \) for every \( n \). But \( \{\lambda_n\} \),
complex numbers, satisfying \( \sum_{n=1}^{\infty} \lambda_n / 2^n = 1 \) and \( |\lambda_n| \leq 1 \) for all \( n \) implies \( \lambda_n = 1 \)
for all \( n \), so \( T_n S_0 x = S_0 x \) for all \( n \). Therefore, \( S_0 x \) is a common fixed point for \( \{T_n\} \)
and, hence, for \( \mathcal{S} \).

**Remark.** As the referee has kindly pointed out, the above proof is similar to a
proof given by R. E. Bruck, Jr., *Properties of fixed-point sets of nonexpansive

**Corollary 1.** A weakly closed convex semigroup of contractions on a separable
Hilbert space has a common fixed point other than 0 if and only if it does not contain
the operator 0.

**Proof.** A Hilbert space satisfies all the hypotheses on \( \mathcal{X} \) in Theorem 1. Also, the
unit ball of operators on a separable Hilbert space is a separable metrizable space in
the weak operator topology, so every semigroup of contractions is separable.

For the next two corollaries let \( \mathcal{S} \) be a weakly closed convex semigroup of
contractions on a separable Hilbert space.

**Corollary 2.** Let \( \mathcal{M} \) denote the set of common fixed points of members of \( \mathcal{S} \); then \( \mathcal{S} \)
contains the orthogonal projection onto \( \mathcal{M} \).

**Proof.** As is well known, \( \|T\| \leq 1 \) and \( Tx = x \) implies \( T^* x = x \) (begin an
orthonormal basis with \( x / \|x\| \) and represent \( T \) with respect to it). Thus, \( \mathcal{M} \) reduces
every operator in \( \mathcal{S} \). Now \( \mathcal{S} \mid \mathcal{M}^\perp \) is a weakly closed convex semigroup of contrac-
tions on \( \mathcal{M}^\perp \). Since the only common fixed point of \( \mathcal{S} \mid \mathcal{M}^\perp \) is \( \{0\} \), Corollary 1
implies that the 0 operator is in $\mathcal{S} \setminus \mathcal{M}$. Let $P \in \mathcal{S}$ be such that $P|\mathcal{M} = 0$; since $P|\mathcal{M}$ is the identity, $P$ is the projection on $\mathcal{M}$.

**Corollary 3.** If $\mathcal{S}$ is not the semigroup consisting only of the identity, then some operator in $\mathcal{S}$ has nontrivial nullspace.

**Proof.** By Corollary 2, if no operator in $\mathcal{S}$ has nullspace, then the set of common fixed points is the entire space.

The next result is a corollary of Theorem 1 in some cases but not in all. The proof, however, is contained in that of Theorem 1.

**Theorem 2.** If $\mathcal{S}$ is a weak operator closed bounded convex set of linear operators on a reflexive space and $0 \in \mathcal{S}$, then 1 is an eigenvalue of every operator $T$ with the property that $S \in \mathcal{S}$ implies $TS \in \mathcal{S}$.

**Proof.** Let $T$ be as stated. By Schauder’s theorem, $TS_0 = S_0$ for some $S_0 \in \mathcal{S}$. Choose $x$ such that $S_0x \neq 0$; then $TS_0x = S_0x$, so 1 is an eigenvalue of $T$.

**Corollary 4.** If $A$ is an injective operator on Hilbert space, and if there is a $k$ such that $||(1 + A)^n|| \leq k$ for every positive integer $n$, then the weakly closed linear span of $\{A^n: n \geq N\}$ is the same for all nonnegative integers $N$.

**Proof.** Let $T = 1 + A$ and let $\mathcal{S}$ be the weakly closed convex hull of $\{T^n: n \geq 1\}$. Since $A$ has no nullspace, $T$ has no fixed points other than 0. By Theorem 2, $0 \in \mathcal{S}$.

Thus, given any weak operator neighborhood $W$ of 0 there is a collection of nonnegative numbers $\{\lambda_j\}_{j=1}^m$ such that $\sum_{j=1}^m \lambda_j = 1$ and $\sum_{j=1}^m \lambda_j T^j \in W$. Then $\sum_{j=1}^m \lambda_j T^j$ has the form $1 + \sum_{j=1}^m \lambda_j p_j(A)$ for suitable polynomials $p_j$ without constant terms. It follows that 1 is in the weak closure of the linear span of $\{A^n: n \geq 1\}$. Thus, $A$ is also in the weak closure of the linear span of $\{A^n: n \geq 2\}$ (multiplication is separately weakly continuous in each variable), and the corollary follows by a trivial induction.

**References**


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