ON THE COMBINATORIAL PROPERTIES OF BLACKWELL SPACES

JAKUB JASIŃSKI

Abstract. Under MA + ~CH (Martin's Axiom and negation of the Continuum Hypothesis) we prove that the intersection of a Blackwell space with the analytic set and the Cartesian product of a Blackwell space and a Borel set do not need to be Blackwell spaces.

1. Introduction. In this paper we present a few examples illustrating the singular behaviour of Blackwell spaces. They shall provide (under MA + ~CH) the negative answers to questions P4, P6, P7 and P8 raised by K. P. S. Bhaskara Rao and B. V. Rao in [1]. The first two were originally answered by W. Bzyl and J. Jasiński in [4]. Here we give a slight generalization of their result (Proposition 1).

On the other hand, W. Bzyl in [3], using the idea presented in [9], proved that questions P4, P6, P7 and a weaker version of P8 have positive answers when restricted to Blackwell spaces with totally imperfect complement in some analytic set.

Let us recall the main definitions. For a metric space $X$ by $2^nA$, we denote the additive Baire classes of Borel subsets of $X$, and by $B(X)$ we denote a $\sigma$-algebra of all Borel subsets of $X$. A Borel measurable mapping $f: X \to Y$, where $Y$ is a metric space, is called a class $2^nA$ if, for all open subsets $U \subset Y$, $f^{-1}(U) \in 2^nA(X)$.

A $\sigma$-algebra of subsets of $X$ is called separable if it is countably generated (c.g.) and separates the points of $X$. Let $X$ be a separable metric space. $X$ is called a Blackwell space if $B(X)$ does not contain a proper separable sub-$\sigma$-algebra. $X$ is called a strongly Blackwell space if any two c.g. sub-$\sigma$-algebras of $B(X)$ with the same atoms coincide. It is clear that a strongly Blackwell space is a Blackwell space. If $A$ is an analytic subset of a Polish space, then $A$ is strongly Blackwell. For this and for other results on Blackwell spaces see K. P. S. Bhaskara Rao and B. V. Rao [1].

2. Basic lemmas. We shall often refer to the well-known result of Silver:

Lemma 1. (MA) If $Z$ is a separable metric space and $|Z| < 2^\omega$, then $B(Z) = P(Z)$.

For the proof see [8, pp. 162, 163]. As pointed out by K. P. S. Bhaskara Rao and B. V. Rao [1, p. 15], Lemma 1 implies the following

Lemma 2. (MA) If $Z$ is a separable metric space with $|Z| < 2^\omega$, then $Z$ is strongly Blackwell.
Lemma 3. If $Y$ is a Blackwell space and $B$ is a Borel subset of the Polish space, then $Y \cup B$ is a Blackwell space.

For the proof see [1, p. 28, 2°].

3. Main propositions. In this section, $X$ will denote a Polish space.

Proposition 1. (MA) Let $B \in \mathcal{B}(X)$ be of cardinality $2^\omega$ and let $Z$ be an uncountable separable metric space of cardinality less than $2^\omega$. If $Z \cap B = \emptyset$, then $Z \cup B$ is a Blackwell space which is not strongly Blackwell.

Proof. By Lemma 2, $Z$ is a Blackwell space, so, by Lemma 3, $B \cup Z$ is also a Blackwell space.

W. Bzyl and J. Jasinski in [4] proved that there exists a Borel set $B_1 \in \mathcal{B}(\mathbb{R}^2)$ and $Z_1 \subseteq \mathbb{R}$ with $|Z_1| = \omega_1$ such that $B_1 \cup Z_1$ is not a strongly Blackwell space. Let $f: B \rightarrow B_1$ be a Borel isomorphism (see [6, p. 450, Theorem 2]) and let $g: Z \rightarrow Z_1$.

By Lemma 1 a mapping $h: B \cup Z \rightarrow B_1 \cup Z_1$, defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in B, \\ g(x) & \text{for } x \in Z, \end{cases}$$

is Borel measurable; hence, $B \cup Z$ is not strongly Blackwell.

A certain part of the next proposition does not require MA so we formulate it separately as

Lemma 4. Let $Y \subseteq X$ be a Blackwell (strongly Blackwell) space and let $Z \subseteq X \setminus Y$. If, for every Borel set $B \in \mathcal{B}(X)$, $B \cap Y = \emptyset$ implies $|B \cap Z| < \omega$, then $Y \cup Z$ is a Blackwell (strongly Blackwell) space.

Proof. We give a proof in case $Y$ is a strongly Blackwell space. Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}(Y \cup Z)$ be c.c.g. $\sigma$-algebras with the same atoms and let $D \in \mathcal{D}$. By [1, p. 23, Proposition 8(5)] it suffices to show that $D \in \mathcal{C}$. Since $Y$ is strongly Blackwell, $\mathcal{C} \uparrow_Y = \mathcal{D} \uparrow_Y$, so there is a set $C \in \mathcal{C}$ such that

$$C \cap Y = D \cap Y. \tag{1}$$

Let $C', D' \in \mathcal{B}(X)$ be such that $C' \cap (Y \cup Z) = C$ and $D' \cap (Y \cup Z) = D$. By (1) the symmetric difference $D' \Delta C' \subseteq X \setminus Y$; hence, $(D' \Delta C') \cap Z \subseteq \omega$ and $|D \Delta C| \leq \omega$, so $D = D \uparrow_D (C \Delta D) \in \mathcal{C}$.

Recall that whenever $A \subseteq X$ is an analytic non-Borel set, then there exist non-empty Borel sets $C_\alpha$, $\alpha < \omega_1$, such that each Borel set $B \in \mathcal{B}(X)$ disjoint with $A$ is covered by countably many $C_\alpha$'s. The sets $C_\alpha$ are called the constituents of a coanalytic set $X \setminus A$ (see [6, p. 499]).

Disjoint sets $X_1, X_2 \subseteq X$ are called Borel-separable if there is a Borel set $B \in \mathcal{B}(X)$ such that $X_1 \subseteq B$ and $X_2 \subseteq X \setminus B$.

1This lemma has been obtained independently by R. M. Shortt and K. P. S. Bhaskara Rao.
PROPOSITION 2. (MA) Let \( A \subseteq X \) be an analytic non-Borel set and let \( \{C_\alpha\}_{\alpha < \omega_1} \) be the constituents of \( X \setminus A \). Whenever \( Z \subseteq X \), \( \omega < |Z| < 2^\omega \) and \( A \cap Z = \emptyset \), then \( A \cup Z \) is not a Blackwell space iff there is an \( \alpha_0 < \omega_1 \) such that \( |C_{\alpha_0} \cap Z| > \omega \) and \( \{(\alpha: C_\alpha \setminus Z \neq \emptyset)\} = \omega_1 \).

**Proof.** "If " part. Let \( Z_1 \subseteq Z \cap C_{\alpha_0} \) be of cardinality \( \omega_1 \). Since
\[
|\{\alpha < \omega_1: C_\alpha \setminus Z \neq \emptyset\}| = \omega_1,
\]
there is a mapping \( g: Z_1 \rightarrow X \setminus (A \cup Z) \) such that \( g(Z_1) \) and \( A \) are disjoint non-Borel-separable.

Define \( f: A \cup Z \rightarrow X \),
\[
f(x) = \begin{cases} 
  x & \text{for } x \in (A \cup Z) \setminus Z_1, \\
  g(x) & \text{for } x \in Z_1.
\end{cases}
\]

By Lemma 1, \( f \) is Borel measurable, but \( f(Z_1) = g(Z_1) \notin \mathcal{B}(f(A \cup Z)) \), so \( A \cup Z \) is not a Blackwell space (see [1 p. 22, Proposition 7(2)]).

"Only if " part. Suppose \( |\{\alpha: C_\alpha \setminus Z \neq \emptyset\}| < \omega_1 \). In this case there is an \( \alpha < \omega_1 \) such that \( Z \cup A = Z \cup (\bigcup_{\beta > \alpha} C_\beta) \cup A \) is Borel, so, by Lemmas 2 and 3, \( A \cup Z \) is a Blackwell space.

In case, for every \( \alpha < \omega_1 \), \( |C_\alpha \cap Z| < \omega_1 \), then, for every Borel set \( B \in \mathcal{B}(X) \), \( B \cap A = \emptyset \) implies \( |B \cap Z| \leq \omega \), so, by Lemma 4, \( Z \cup A \) is a Blackwell space.

**Corollary 1.** (MA) Let \( A \) and \( Z \) be as in Proposition 2. If \( A \) and \( Z \) are Borel separable, then \( A \cup Z \) is not a Blackwell space.

**Corollary 2.** (MA + CH) There exists a Blackwell space \( Y \) and an analytic set \( A \subseteq X \) such that \( Y \cap A \) is not a Blackwell space.

**Proof.** Let \( Y = B \cup Z \) where \( B \in \mathcal{B}(X) \), \( |B| = 2^\omega \), \( Z \subseteq X \), \( \omega < |Z| < 2^\omega \) and \( B \cap Z = \emptyset \). By Lemmas 2 and 3, \( Y \) is a Blackwell space. Let \( A_1 \subseteq B \) be an analytic non-Borel set and take \( A = A_1 \cup (X \setminus B) \). \( A \cap Y = A_1 \cup Z \) which is not Blackwell by Corollary 1.

**Proposition 3.** (MA) If \( Z \subseteq X \) and \( \omega < |Z| < 2^\omega \), then \( Z \times B \) is not a Blackwell space, where \( B \) is an uncountable Borel subset of some Polish space.

We shall precede the proof with two lemmas. The first one follows by Lemma 1 from [2, Theorem 3].

**Lemma 5.** (MA) Let \( \mathcal{N} \) be a set of irrational numbers. If \( Z \subseteq \mathcal{N} \) with \( |Z| < 2^\omega \), then \( B \in \mathcal{B}(Z \times \mathcal{N}) \) iff there is an \( \alpha < \omega_1 \) such that for every \( z \in Z \) the section \( B_z = \{y: (z, y) \in B\} \in \Sigma_\alpha(\mathcal{N}) \).

Lemma 5 implies the following

**Lemma 6.** (MA) Let \( Z \) and \( B \) be as in Proposition 3. A mapping \( h: Z \times B \rightarrow Y \), where \( Y \) is a separable metric space, is Borel measurable iff there is an \( \alpha < \omega_1 \) such that for every \( z \in Z \) a restricted mapping \( h \upharpoonright \{z\} \times B \) is of class \( \Sigma_\alpha \).
Proof. Apply the isomorphism theorem [6, p. 450, Corollary 1c].

Proof of Proposition 3. By the well-known theorem of Hausdorff [5], \( \mathcal{N} \) can be decomposed into \( \omega_1 \) disjoint uncountable sets of class \( \Sigma_1(\mathcal{N}) = \bigcup_{\alpha < \omega_1} E_\alpha \).

Let \( Z_1 \subset Z \) be of cardinality \( \omega_1 \). By Lemma 1, \( Z_1 \times B \in \mathcal{B}(Z \times B) \), so it suffices to prove that \( Z_1 \times B \) is not a Blackwell space (see [1, p. 28, 1°]). Let \( Z_1 = \{ z_\alpha : \alpha < \omega_1 \} \). There is a \( \gamma < \omega_1 \) such that for each \( \alpha < \omega_1 \) there is a Borel measurable function \( f_\alpha : B \to E_\alpha \) of class \( \Sigma_\gamma \) (see [6, p. 450, Theorem 2]). By Lemma 6, a mapping \( h: Z_1 \times \mathcal{N} \to \mathcal{N} \) defined by \( h(z_\alpha, x) = f_\alpha(x) \) is Borel measurable, but the inverse mapping is not, since \( Z_1 \times B \) is not Borel [6, p. 489, Theorem 1]. Hence, by [1, p. 22, Proposition 7 (2)], \( Z_1 \times B \) is not a Blackwell space.

References

5. F. Hausdorff, Summen von \( \mathcal{F}_1 \) Mengen, Fund. Math. 26 (1936), 248.
7. E. Marczewski, Characteristic function of the sequence of sets and some of its applications, Fund. Math. 31 (1938), 207–223.

Institute of Mathematics, University of Gdansk, Ul. Wita Stwosza 57, 80–952 Gdansk, Poland