ON THE KERNEL OF A MARKOV PROJECTION ON \( C(X) \)

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ABSTRACT. Let \( X \) be a compact metric space and \( L \) a closed linear subspace of \( C(X) \), the real valued continuous functions on \( X \). We give necessary and sufficient conditions of an algebraic nature for \( L \) to be the kernel of a Markov projection \( P \) on \( C(X) \). We also characterize compact spaces for which our result holds as those for which the Borsuk-Dugundji simultaneous extension theorem holds.

1. Introduction. A projection \( P \) on \( C(X) \) is Markov if \( Pe = e \) (where \( e \) is the unit function) and \( P \geq 0 \), i.e., \( f \geq 0 \) implies \( Pf \geq 0 \). If \( P^* \) is the adjoint of \( P \) and \( \delta \), the Dirac measure at \( x \), let \( p_x = P^*\delta \), so that \( p_x \) is a probability measure, and for \( f \in C(X) \) we have \( Pf(x) = \int f \, dp_x \). Let \( P \) be the set of Borel probability measures on \( X \), a compact convex set in \( C(X)^* \), relative to the weak*-topology. Then \( P^*(P) \) is a compact convex set, and each extreme point \( m \) has the form \( p_x \) for some \( x \in X \)—just note that \( p_x^{-1}(m) \) is a convex compact subset of \( P \), and hence contains an extreme point, which is \( \delta_x \) for some \( x \in X \) [4, p. 34].

If \( m \) is a positive Borel measure, \( \text{supp} \, m \) denotes the closed support set of \( m \), and if \( m \) is any Borel measure, \( \text{supp} \, m \) is defined as \( \text{supp} \, |m| \). If \( P \) is a Markov projection, we define \( \text{supp} \, P = \text{closure} \cup \{ \text{supp} \, m : P^*m = m \} \). (Note that \( m \in \text{ran} \, P^* \) iff \( P^*m = m \).)

The structure of \( P \) is pretty well known. Birkhoff [1] and Kelley [3] characterized those \( P \) for which \( \text{ran} \, P \) is an algebra by the following properties: for each \( x \in X \), \( p_x \) is an extreme point of \( P^*(P) \), and for each \( f \in C(X) \), \( Pf \) is constant on \( \text{supp} \, p_x \). Moreover, \( P \) satisfies the averaging identity \( P(fPg) = PfPg \). Lloyd [5] showed that if \( P \) is an arbitrary Markov projection, then \( Pf \) is constant on \( \text{supp} \, p_x \), whenever \( p_x \) is an extreme point of \( P^*(P) \). It follows easily that the natural restriction of \( P \) to a projection on \( C(\text{supp} \, P) \) satisfies the Birkhoff-Kelley conditions. Later Lloyd and Seever found the following identity for all Markov projection: \( P(fPg) = P(PfPg) \) ([6 and 7], see also [9]).

This formula may be rewritten as \( 0 = P((f - Pf)Pg) \), i.e., \( f_0 \in \ker \, P \) and \( g_0 \in \text{ran} \, P \), then \( f_0g_0 \in \ker \, P \). This condition is not quite strong enough to characterize the kernel of a Markov projection, so we note a natural property of such projections, namely if \( f \geq 0 \), then \( Pf = 0 \) iff \( f \) vanishes on \( \text{supp} \, P \). This is an obvious consequence of the fact that for \( x \in X \), \( p_x \) is a probability measure. Thus, if \( P \) is a
Markov projection we have

(1) \( \text{ker} \, P + \text{ran} \, P = C(X) \),
(2) \( (\text{ran} \, P)(\text{ker} \, P) \subset \text{ker} \, P \),
(3) \( I = \{ f : P^2 f = 0 \} \) is an ideal in \( C(X) \).

(Note that if \( m \) is a nonpositive Borel measure with \( m(e) = 1 \), and we define \( P \) by
\( Pf(x) = m(f) \) for all \( f \in C(X) \), then (1) and (2) hold, but not (3).)

Our main result is

**Theorem.** Let \( X \) be compact metric, \( L \) a proper closed linear subspace of \( C(X) \), and \( M = \{ f : fL \subset L \} \). If

(a) \( L + M = C(X) \), and
(b) \( I = \{ f : f^2 \in L \} \) is an ideal,

then there exists a Markov projection \( P \) on \( C(X) \) such that \( L = \text{ker} \, P \) and \( \text{ran} \, P \subset M \).

2. Preliminaries. Throughout, \( L \) will be a closed subspace of \( C(X) \), the real valued continuous functions on \( X \), and \( M \) and \( I \) are as defined in the Theorem. In this section we study the structure of \( I \) after we give some definitions.

Let \( L^+ = \{ m \in C(X)^* : m(f) = 0 \text{ for all } f \in L \} \), and let \( (L^+)\_1 \) be the closed unit ball in \( L^+ \), a compact convex set in the weak*-topology. Note that \( f \in L \) iff \( m(f) = 0 \) for all \( m \in L^+ \) (by Hahn-Banach). Obviously, \( f \in M \) iff \( f \, dm \in L^1 \) for all \( m \in L^+ \), so \( M = \{ f \in C(X) : fL^+ \subset L^+ \} \). Further, \( f \in M \) iff \( f \) is constant on \( \text{supp} \, m \) for each extreme point \( m \in (L^+)\_1 \) [4, pp. 35-36]. We also define \( Z(I) = \{ f^{-1}(0) : f \in I \} \) and \( supp \, L^+ = \text{closure} \bigcup \{ \text{supp} \, m : m \in L^1 \} \). If \( f \in C(X) \) and \( A \subset X \), then \( f\_A \) is the restriction of \( f \) to \( A \), and \( L\_A = \{ f\_A : f \in L \} \).

2.1 Remark. \( Z(I) \subset \text{supp} \, L^+ \).

Proof. If \( x \notin \text{supp} \, L^+ \), then by complete regularity there exists \( f \in C(X) \) which vanishes on \( \text{supp} \, L^+ \), but \( f(x) \neq 0 \). Then \( f^2 \in L \), so \( f \in I \) and \( x \notin Z(I) \).

2.2 Proposition. The following are equivalent:

(a) \( I \) is an ideal,
(b) \( Z(I) = \text{supp} \, L^+ \).

Proof. (b) implies (a). We show \( f \in I \) iff \( \text{supp} \, L^+ \subset f^{-1}(0) \), so that \( I \) is the ideal \( \{ g : \text{supp} \, L^+ \subset g^{-1}(0) \} \). If \( f \in I \), then (b) implies \( \text{supp} \, L^+ \subset f^{-1}(0) \). If \( \text{supp} \, L^+ \subset f^{-1}(0) \), then for all \( m \in L^+ \), \( 0 = m(f^2) \), so \( f^2 \in L \) and \( f \in I \).

(a) implies (b). To show \( \text{supp} \, L^+ \subset Z(I) \), let \( f \in I \) and \( m \in L^+ \). Let \( m = m^+ - m^- \) be the Lebesgue decomposition with \( m^+ \) supported by the Baire set \( A \) and \( m^- \) supported by \( X \setminus A \). Let \( g_n \in C(X) \) with \( 1 \geq g_n \geq 0 \) and \( g_n \to 1_A \) \(|m|\)-a.e. Now \( f g_n \in I \) so \( f^2 g_n^2 \in L \), and

\[
\int f^2 \, dm^+ = \int f^2 1_A \, dm = \lim \int f^2 g_n^2 \, dm = 0
\]

since \( m \in L^+ \). Likewise \( \int f^2 \, dm^- = 0 \), so \( f^2 = 0 \) \(|m|\)-a.e. By continuity, \( \text{supp} \, m \subset f^{-1}(0) \), and since \( m \) is arbitrary, \( \text{supp} \, L^+ \subset f^{-1}(0) \).

2.3 Proposition. If \( M + L = C(X) \) and \( m \) is an extreme point of \( (L^+)\_1 \), then \( m(e) \neq 0 \).
Proof. Let $S = \text{supp } m$. (Since $L$ is proper, $m \neq 0$.) If $f \in M$ then $f$ is constant on $S$. By hypothesis $C(S) = L_S + M_S$. But then $C(S) = L_S + \text{constants}$, so if $g \in C(S)$ we have $g = h + ce$ with $h \in L_S$ and $c$ constant, whence $m(g) = m(h) + cm(e) = 0 + cm(e)$. If $m(e) = 0$, then $m = 0$, which is impossible.

2.4 Proposition. If $L + M = C(X)$, then (a) and (b) in 2.2 are equivalent to
(c) $I \subset M$.

Proof. (b) implies (c). If $f \in I$, then $f$ is constant (in fact, 0) on $\text{supp } m$ whenever $m \in L^\perp$. Hence, $f \in M$ [4, pp. 35–36].

(c) implies (b). By 2.1 we always have $Z(I) \subset \text{supp } L^\perp$. Conversely, if $f \in I$, then (c) implies $f$ is constant on $\text{supp } m$ whenever $m$ is extreme in $(L^\perp)_1$. But since $f^2 \in L$ as well, $m(f^2) = 0$. Since $m(e) \neq 0$, $f^2$ must be 0 on $\text{supp } m$. It is an easy consequence of Krein-Milman that sets of the form $\text{supp } m$, with $m$ extreme in $(L^\perp)_1$, are dense in $\text{supp } L^{-1}$, so $\text{supp } L^{-1} \subset f^{-1}(0)$.

2.5 Proposition. Let $I_0 = \{ f \in C(X) : f \in L \text{ and } f^2 \in L \}$. If $I$ is an ideal, then $I = I_0$, and hence $I \subset L \cap M$, provided $L + M = C(X)$.

Proof. Clearly, $I_0 \subset I$. If $I$ is an ideal, then $Z(I) = \text{supp } L^\perp$, by 2.2, so if $f \in I$, then $0 = m(f) = m(f^2)$ for all $m \in L^\perp$, whence $f \in L$ as well as $f^2 \in L$. Thus $f \in I_0$.

2.6 Remark. Propositions 2.2 and 2.4 remain true if $I$ is replaced by $I_0$. This fact is not needed below, and we omit the easy proof. In §4 we give some examples on the relation between $I$ and $I_0$.

3. Proof of Theorem. (i) Let $Z = Z(I)$. By 2.4, hypotheses (a) and (b) of the Theorem imply $Z = \text{supp } L^\perp$. We now prove $I = L \cap M$. By 2.5 we already have $I \subset L \cap M$. Conversely, if $f \in L \cap M$, then $f$ is constant on $\text{supp } m$ for $m$ extreme in $(L^\perp)_1$, while $m(f) = 0$ because $f \in L$. Since by 2.3 $m(e) \neq 0$, we have $f = 0$ on $\text{supp } m$. It follows that $L^\perp \subset f^{-1}(0)$, so $f \in I$.

(ii) Since $C(X) = L + M, I = L \cap M$, and $Z = Z(I)$, we have $C(Z) = L_Z \oplus M_Z$. Thus, there exists a projection $Q$ on $C(Z)$ whose kernel is $L_Z$ and whose range is $M_Z$. If $e_z$ is the restriction of $e$ to $Z$, then clearly $Qe_z = e_z$, and it remains to show that $Q \geq 0$ (and then that $Q$ extends to a Markov projection $P$ on $C(X)$).

(iii) First we show that because (1) $\text{ran}(Q) \text{ker}(Q) \subset \text{ker}(Q)$ and (2) $\text{ran}(Q)$ is an algebra, we have $Q(fg) = QfQg$ for all $f$ and $g$ in $C(Z)$.

$$Q(fg) = Q((f - Qf + Qf)Qg) = Q((f - Qf)Qg + Q(QfQg))$$

$$= 0 + QfQg.$$  

(iv) Secondly, if $f \geq 0$ and $Qf = 0$, then $f = 0$ on $Z$. Let $F \in C(X)$ satisfy $F \geq 0$ and $F_Z = f$. Since $f \in L_Z$, there exists $G \in L$ with $G_Z = f$, i.e., $G_Z = F_Z$. If $m \in L^\perp$, then $\text{supp } m \in Z$, so $m(F) = m(G) = 0$, so $F \in L$. Since $F \geq 0$, we have $F^{1/2} \in I \subset M$. Since $M$ is an algebra, $F \in M$, i.e., $F \in L \cap M = I$, so $f = F_Z = 0$.

(v) Finally, suppose there exists $f \in C(Z)$ with $f \geq 0$, but $Qf(x) < 0$ for some $x$. The set $V = \{ y : Qf(y) < 0 \}$ is open in $Z$ relative to the topology generated by the
subalgebra \( M_z = Q(C(Z)) \), which is completely regular, but not Hausdorff. Hence, there exists \( g \in M_z \) such that \( g(x) = 1 \), \( g = 0 \) off \( V \), and \( 0 \leq g \leq 1 \). Let \( h = gf \). Then \( h \geq 0 \), and, by (iii), \( Qh = Q(gf) = Q((Qg)f) = QgQf = gQf \). So \( Qh(x) = Qf(x) < 0 \), \( Qh \leq 0 \) on \( V \), and \( Qh = 0 \) off \( V \). Let \( k = h - Qh \geq 0 \). Then \( Qk = 0 \), so, by (iv), \( k = 0 \) on \( Z \), i.e., \( h = Qh \). But this is impossible since \( h(x) \geq 0 \) and \( Qh(x) < 0 \). (The last three lines were inspired by a homework paper of graduate student Pengyuan Chen.)

(vi) We now show that \( Q \) extends to a Markov projection on \( C(X) \). Since \( X \) is compact metric (and this is the only time metrizability is used) there exists a simultaneous extender, i.e., a positive linear map \( E: C(Z) \rightarrow C(X) \) such that, for \( x \in Z \), \( f(x) = Ef(x) \), and also \( Ee_Z = e_X = e \). (See the Borsuk-Dugundji theorem in [8, p. 365].) We define \( P \) by \( Pf(x) = E(Q(fz))(x) \). It is easy to check that \( P \) is a Markov projection, and we must show that \( L = \ker P \) and \( \text{ran } P \subseteq M \).

(vii) To show \( L \subseteq \ker P \), if \( f \in L \), then \( fz \in L_Z \), so \( Pf = E(Q(fz)) = E(0) = 0 \). To show \( \ker P \subseteq L \), suppose \( 0 = Pf = E(Q(fz)) \). If \( m \in C(X)^* \) and \( \text{supp } m \subseteq Z \), let \( m_Z \) be \( m \) considered as an element of \( C(Z)^* \), so for \( g \in C(Z) \), \( m(g) = m_Z(g_Z) \), and for \( g \in C(Z) \), \( m_Z(g) = m(Eg) \). Then \( m \in L^\perp \) iff \( m_Z \in (L_Z)^\perp \). Since \( L_Z = \ker Q \) and \( Q \) is a projection, \( (L_Z)^\perp = \text{ran } (Q^*) \), so \( m \in L^\perp \) iff \( Q^*m_Z = m_Z \). Hence, for all \( m \in L^\perp \),

\[
\begin{align*}
m(f) &= m_Z(fz) = Q^*m_Z(fz) = Q_Z(Q(fz)) = m(E(Qfz)) \\
&= m(Pf) = m(0) = 0.
\end{align*}
\]

It follows that \( f \in L \).

(viii) To show \( \text{ran } P \subseteq M \), since \( L = \ker P \) and \( P \) is a Markov operator, property (2) of the introduction says \( (\text{ran } P)L \subseteq L \).

4. Examples. We assumed metrizability of \( X \) only in order to invoke the Borsuk-Dugundji extension theorem. The following rather surprising result shows that the extension theorem is necessary as well as sufficient.

4.1 Proposition. If \( X \) is a compact Hausdorff space, the following are equivalent:

(a) If \( Z \) is a closed subset, there exists a Markov extension operator \( E: C(Z) \rightarrow C(X) \).

(b) The result of our main theorem holds for \( C(X) \).

Proof. We already know that (a) implies (b). Conversely, suppose (b) holds. If \( Z \) is closed in \( X \), let \( L = \{ f : f_z = 0 \} \) be an ideal. Then \( L = L \), so \( L \) is an ideal, and \( M = C(X) \), so \( M + L = C(X) \). By (b) there exists a Markov projection \( P \) with \( \ker P = L \). Now \( \text{ran } P^* = L^\perp = C(Z)^* \), the space of regular Borel measures on \( Z \). That is, if \( m \in C(X)^* \) and \( \text{supp } m \subseteq Z \), then \( P^*m = m \). We define the extension operator \( E \) as follows: if \( f \in C(Z) \), let \( f_1 \) be any norm-preserving extension of \( f \) to an element of \( C(X) \), and let \( Ef = Pf_1 \). To show \( E \) is well defined, suppose \( f_2 \) is any other extension of \( f \) to an element of \( C(X) \). If \( x \in X \), then \( \text{supp } p_x \subseteq Z \), so \( Pf_1(x) = Pf_2(x) = \int f \, dp_x \). To show \( E \) is an extension operator, i.e., \( (Ef)_Z = f \), let \( x \in Z \). Then \( P^*\delta_x = \delta_x \), so \( Ef(x) = Pf_1(x) = P^*\delta_x(f_1) = f_1(x) = f(x) \). This completes the proof.

Remark. The extension property fails for \( X = \beta N \) and \( Z = \beta N \setminus N \) [8, p. 375].
4.2 Example. We give an example to show that the hypothesis $L + M = C(X)$ is really needed for Propositions 2.3 and 2.4. Let $X = \{1, 2, 3, 4\}$ with the discrete topology, so that $C(X)$ is essentially $R^4$. For simplicity we identify $f \in C(X)$ with its values $(a, b, c, d)$. Let

$$L = \{(a, -a, b, b) : a, b \in R\},$$

so $L^\perp$ is the span of the measures whose values at points are $(1, 1, 0, 0)$ and $(0, 0, 1, -1)$. Now $M = \{(a, a, b, b) : a, b \in R\}$ so $M + L \neq C(X)$. $I = I_0 = \{(0, 0, a, a) : a \in R\}$, which is not an ideal. However, $I \subset M$, so 2.4 fails. Further, $m = (0, 0, \frac{1}{2}, -\frac{1}{2})$ is an extreme measure in $(L^\perp)_1$, but $m(e) = 0$, so 2.3 fails.

We now mention without details some other simple examples we have. (i) $L + M = C(X)$, $I_0$ is not an ideal, $I \neq I_0$, $I \not\subset M$; (ii) $L + M = C(X)$, $I_0$ is an ideal, $I$ is not; (iii) $L + M \neq C(X)$, $I_0$ is an ideal, $I$ is not, and $I \not\subset M$.

5. Remarks. I do not know whether our result is valid in noncommutative $C^*$-algebras. It is known that for unital $C_0$-algebras, the identity $P(PaPb) = P(aPb)$ holds, where multiplication is the Jordan product [10, Lemma 1.1].

From [2] it is clear that contractive projections are more complicated than Markov projections, and it is not generally true that $(\text{ran } P)(\text{ker } P) \subset \text{ker } P$. In fact, if $f \in C_c(X)$ (the complex continuous functions) and $m$ is extreme in $(L^\perp)_1$, where $L = \text{ker } P$, then on $\text{supp } m$, $Pf$ is a constant times the Radon-Nikodym derivative $d|m|/dm$. (If $P$ is Markov, then $|m| = \pm m$, so $Pf$ is constant on $\text{supp } m$.) It is an easy consequence of this that $(\text{ran } P)(\text{ran } P)^* \subset \text{mult } P$, or, equivalently, the identity $P(Pf(Pg)^*Ph) = P(f(Pg)^*Ph)$—the bar stands for complex conjugation. In fact, this is proved for general $C^*$-algebras in [11, Corollary 3].

Finally, in view of Proposition 4.1, it would be interesting to find characterizations—topological or analytic—of compact spaces for which the extension theorem holds. See [8] for references.

I am grateful to A. Iwanik for pointing out that our result fails if $L$ is not a proper subspace of $C(X)$.

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