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ISOMETRIES OF THE DISC ALGEBRA

MOHAMAD EL-GEBEILY AND JOHN WOLFE

ABSTRACT. The linear isometries $u: A \to A$ of the disc algebra A into itself are completely described. Such isometries u must be one of two distinct types. The first type is $uf = \psi \cdot f(\phi)$, where $\psi \in A$ and $\phi \in H^\infty$ satisfy certain described conditions. The second type is $uf = E(\psi \cdot f(\phi))$, where $\phi: Q \to T$ is any continuous function from a closed zero measure subset Q of the unit circle T onto itself, $\psi \in C(Q)$ is unimodular, and $E: Y \to A$ is a norm 1 extension operator, where $Y = \{\psi \cdot f(\phi): f \in A\} \subset C(Q)$. Isometries of $C(K)$ spaces into the disc algebra are also described.

1. Introduction. A linear operator $u: X \to X$ on a Banach space X is an *isometry* if $\|ux\| = \|x\|$ for each $x \in X$. The isometries of most of the well-known Banach spaces have been described. The isometries of $C(K)$ spaces were described by Banach and Stone (onto case) and Holsztynski [**12**] (into case). Isometries of $L^p(\mu)$ spaces and H^p were worked out by Lamperti [**16**] and Forelli [**9**], respectively. The onto isometries of the disc algebra A and H^∞ were determined by de Leeuw, Rudin, and Wermer [**17**]. Several further papers dealing with isometries on various spaces are listed in the bibliography. In this paper we describe the isometries of the disc algebra A into itself. This answers a question raised by Phelps [**30**, p. 354].

Our notation follows Rudin [**26**] and Hoffman [**11**]. We use D for the open unit disc in the complex plane, $\overline{D} = \{z: |z| \leq 1\}$, and $T = \{z: |z| = 1\}$. We use $C(T)$, $C(\overline{D})$, and $C(K)$ to denote the sup norm Banach spaces of continuous complex valued functions on T, $\overline{D}$, or a general compact Hausdorff space K, respectively. The disc algebra $A = \{f \in C(\overline{D}): f \text{ is analytic on } D\}$, and H^∞ is the sup norm Banach space of bounded analytic functions on D. Lebesgue measure on T is denoted by m. We identify A in its natural way as a subspace of $C(T)$.

We begin by discussing Propositions 1 and 2 which describe two types of isometries of the disc algebra into itself. The main result (Theorem A) is that any isometry on A must be of the form described in either Propositions 1 or 2. Theorem B describes the isometries of $C(K)$ spaces into A.

These results are proved in §2. §3 contains a few further remarks and open questions.

PROPOSITION 1. *Suppose $\phi \in H^\infty$ and $\|\phi\| \leq 1$, where $\phi = h_1/h_2$ with $h_1, h_2 \in A$. Let $S = \{t \in T: h_2(t) = 0\}$. Suppose $\psi \in A$ and $\psi(s) = 0$ for $s \in S$.*

(a) *Then $uf = \psi \cdot f(\phi)$ defines a bounded linear operator from A into A.*

(b) *The operator u is an isometry $\Leftrightarrow$ $\|\phi\| = \|\psi\| = 1$, and there is a closed set Q in T such that $Q \cap S = \emptyset$, $\phi(Q) = T$ and $|\psi(q)| = 1$ for all $q \in Q$.*

An isometry $u: A \to A$ of the form described in Proposition 1(b) will be called a *Type* 1 *isometry* on A.

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McDonald [**33**, Proposition 1.1] shows that isometries as in part (b) are precisely the ones satisfying $(u1)u(fg) = (uf)(ug)$.

Type 1 isometries are quite natural and expected in this situation, since for many function spaces X (e.g., on H^p or L^p) all isometries are of the form $uf = \psi \cdot f(\phi)$, where the conditions on ψ and ϕ depend on the nature of X.

A wide variety of allowable functions ϕ and ψ, for which $uf = \psi \cdot f(\phi)$ is an isometry of A, can be imagined. For example, if $\phi_1 \in A$ is a "Riemann map" of $\overline{D}$ to the quarter annulus $\{z\colon 1/2 \leq z \leq 1 \text{ and } 0 \leq \arg z \leq \pi/2\}$, then some proper arc Q of T is mapped onto $\{z\colon |z| = 1 \text{ and } 0 < \arg z < \pi/2\}$. Thus $uf = \psi f(\phi)$ is an isometry, where $\phi = (\phi_1)^4$ and $\psi \in A$ is any norm 1 function which is unimodular on Q.

We also note that $\phi \in H^\infty$ is of the form $\phi = h_1/h_2$, where $h_1, h_2 \in A \Leftrightarrow$ the radial limit of ϕ is continuous off some closed subset of T with measure zero.

The next proposition is trivial.

PROPOSITION 2. *Let Q be a closed subset of T of measure zero, let $\phi\colon Q \to T$ be any continuous onto map, and let $\psi \in C(Q)$ satisfy $|\psi(q)| = 1$ for each $q \in Q$. Define the subspace Y of $C(Q)$ by $Y = \{\psi \cdot f(\phi)\colon f \in A\}$. Let $E\colon Y \to A$ be a linear extension operator with $\|E\| = 1$, i.e., E is a bounded linear operator of norm 1 such that, for each $f \in Y$ and $q \in Q$, $E(f)(q) = f(q)$. Then $uf = E(\psi \cdot f(\phi))$ defines an isometry of A into A.*

An isometry $u\colon A \to A$ of the form described in Proposition 2 will be called a *Type* 2 *isometry* on A.

We are now in a position to state our main result.

THEOREM A. *Any isometry of A is either of Type* 1 *or of Type* 2, *i.e., of the form described in either Proposition* 1 *or Proposition* 2.

One reason for the existence of what we have called Type 2 isometries on A is that A contains isometric copies of $C(T)$ (see Pelczynski [**21**]). This second type of isometry previously appeared (in the context of Banach spaces of the type $C(K)$, K a compact Hausdorff space) in Holsztynski [**12**] (see also Proposition 1 of Pelczynski [**23**]).

In order to illuminate the nature of Type 2 isometries, we make the following observations.

We note that for any closed subset Q of T of measure zero there are many norm 1 extension operators $E\colon C(Q) \to A$ as shown by Pelczynski [**21**] and Michael and Pelczynski [**18**] (also see Rudin [**25**], Carleson [**6**], and Bishop [**4**]).

We also note that if $Q \subset T$ has measure zero then there exist continuous maps $\phi\colon Q \to T$ of Q onto $T \Leftrightarrow Q$ is uncountable (e.g., Q is a homeomorph of the Cantor set). This argument goes as follows: (1) $m(Q) = 0$ implies Q is totally disconnected (it cannot contain intervals); (2) Q uncountable implies that it contains a homeomorph of the Cantor set which must be a retract of Q; (3) there are well-known maps of the Cantor set onto T.

Thus we see that Type 2 isometries exist in profusion. Unfortunately, it seems that Type 2 isometries can never be described very explicitly, since, firstly, the maps $\phi\colon Q \to T$, in the few cases in which they are explicit, are rather ugly, and, secondly, the extension operators $E\colon C(Q) \to A$, which are constructed by Michael and Pelczynski using a limiting process, always seem to be illusive.

Our final result describes the isometries of a $C(K)$ space into A.

THEOREM B. *Let K be a compact metric space, and let $u\colon C(K) \to A$ be an isometry. Then u is Type 2. More precisely, $uf = E(\psi \cdot f(\phi))$, where Q is a closed subset of T of measure zero, $\phi\colon Q \to K$ is continuous and onto, $\psi \in C(Q)$ satisfies $\psi(q) = 1$ for all $q \in Q$, and $E\colon Y \to A$ is a norm 1 extension operator, where $Y = \{\psi \cdot f(\phi)\colon f \in C(K)\}$.*

2. Proof of the results. Our basic tool for proving Theorems A and B is the following proposition.

PROPOSITION 3. *Let $u\colon A \to A$ be an isometry. Then there exist a closed subset Q of T, a continuous map $\rho\colon Q \to T$, and a continuous onto map $\phi\colon Q \to T$ such that $\rho(q)u(g)(q) = g(\phi(q))$ for all $g \in A$ and $q \in Q$.*

We identify A^* as a quotient space of $M(T)$, where $M(T) = C(T)^*$ denotes the space of regular Borel measures on T. The notation $B(A^*)$, $B(C(T)^*)$, and $\operatorname{ext} B(A^*)$ and $\operatorname{ext} B(C(T)^*)$ denote, respectively, the unit balls and extreme points of the unit balls of A^* and $C(T)^* = M(T)$. For $t \in T$, $\delta_t \in A^*$ denotes the point evaluation. Since $t \to \delta_t$ is a homeomorphism of T into A^* equipped with the weak* topology, we identify T with the subset $\{\delta_t\colon t \in T\}$ of A^*.

The following proof is a straightforward adaptation of Pelczynski's proof [**23**, Proposition 1] of a result of Holsztynski [**12**] on isometries of $C(K)$ spaces.

PROOF OF PROPOSITION 3. We first establish that for each $t \in T$ the set $K_t \neq \emptyset$, where $K_t = ((u^*)^{-1}\delta_t) \cap \operatorname{ext} B(A^*)$. First of all, $\tilde{K}_t = ((u^*)^{-1}\delta_t) \cap B(A^*) \neq \emptyset$ because u is an isometry and, thus, $u^*(B(A^*)) = B(A^*)$. But $\tilde{K}_t$ (using the terminology of §V.8 of Dunford and Schwartz [**7**]) is a weak* compact extremal subset of $B(A^*)$, so it has extreme points which will also be extreme points of $B(A^*)$, which shows that $K_t \neq \emptyset$.

Now for each $\lambda \in T$ let $Q_\lambda = (u^{*-1}(\lambda T)) \cap T$ and let $Q = \bigcup_{\lambda \in T} Q_\lambda$. Define $\rho\colon Q \to T$ by $\rho(q) = \lambda^{-1}$ if $q \in Q_\lambda$, and define $\phi\colon Q \to T$ by $\phi(q) = \rho(q)u^*(\delta_q)$. The last paragraph shows that ϕ maps Q onto T (since $\operatorname{ext} B(A^*) = \{\alpha\delta_t\colon t \in T,\ |\alpha| = 1\}$). Also, by definition, for $q \in Q$ and $g \in A$,

$$\rho(q)(ug)(q) = \rho(q)(u^*(\delta_q))g = \delta_{\phi(q)}g = g(\phi(q)).$$

To see that Q is closed and ρ is continuous, let F be a closed subset of T. Then

$$\rho^{-1}(F) = \bigcup_{\lambda \in F} Q_{\lambda^{-1}} = \bigcup_{\lambda \in F} (u^{*-1}(\lambda^{-1}T) \cap T)$$
$$= \left(u^{*-1}\left(\bigcup_{\lambda \in F} \lambda^{-1}T\right)\right) \cap T = (u^{*-1}(F^{-1} \times T)) \cap T$$

is weak* closed since

$$F^{-1} \times T = \{\lambda^{-1}t\colon \lambda \in F,\ t \in T\}$$

is closed and u^* is weak* continuous. Then ρ is continuous and $Q = \rho^{-1}(T)$ is closed. This proves Proposition 3.

PROOF OF PROPOSITION 1. To see $uf = \psi \cdot f(\phi)$ is bounded from A to A observe that, for $f \in A$, uf is analytic on D and, for $z \in D$, $|uf(z)| \leq \|\psi\|\,\|f\|$, so u is bounded from A into H^∞. To get $u(A) \subset A$ we need $u(z^n) \in A$ for each $n \geq 1$.

But $uz^n = \psi h_1^n/h_2^n$, and for $t \in T\backslash S$ this is certainly continuous. If $t_0 \in S$ then

$$\lim_{t\to t_0} |(uz^n)(t)| \le \lim_{t\to t_0} |\psi(t)| \overline{\lim_{t\to t_0}} |\phi^n(t)| = 0,$$

since $\phi \in H^\infty$ and $\psi(t_0) = 0$. This proves (a).

For (b) we first note that, since $Q \cap S = \emptyset$, although $\phi \in H^\infty$, its radial limit is continuous on $T - S$ so that $\phi(Q)$ makes sense. The proof that a Type 1 operator is an isometry is straightforward. Conversely, if $uf = \psi \cdot f(\phi)$ is an isometry, then the set Q is obtained from Proposition 3. Taking g in Proposition 3 to be 1 gives $|\psi(q)| = |\rho(q)^{-1}| = 1$ for $q \in Q$ (hence, $Q \cap S$ is void). Taking g to be z gives the desired conclusion $\phi(Q) = T$, and the rest of the proof is apparent.

PROOF OF THEOREM A. Given the isometry $u\colon A \to A$ let $Q, \rho\colon Q \to T$ and $\phi\colon Q \to T$ be as given in Proposition 3, i.e., for $q \in Q$, $\rho(q)(uf)(q) = f(\phi(q))$. We will show that if $m(Q) > 0$ then u is a Type 1 isometry, and if $m(Q) = 0$ then u is a Type 2 isometry.

So first assume $m(Q) > 0$. The proof of this case is similar to the proof of Theorem 1.1 of McDonald [**33**].

Letting $f = 1$ we get, for $q \in Q$, $\rho(q)(u1)q = 1$ or $u1 = 1/\rho$ on Q. Thus,

$$(1) \qquad uf(q) = (u1)(q)f(\phi(q)) \quad \text{for } f \in A \text{ and } q \in Q.$$

Next we establish

$$(2) \qquad (u1)(u(fg)) = (uf)(ug) \quad \text{for } f, g \in A.$$

For from (1), for $q \in Q$,

$$(u1)(q)u(fg)(q) = (u1(q))^2 f(\phi(q))g(\phi(q)) = (uf)(q)(ug)(q).$$

Thus, (2) holds on Q. But $m(Q) > 0$ and the functions involved are in A, so (2) holds on $\overline{D}$.

It follows immediately from (2) that

$$(3) \qquad (u1)^{n-1}u(z^n) = (uz)^n \quad \text{for } n \ge 1.$$

Now define $\phi_1(\xi) = uz(\xi)/u1(\xi)$. We now show that ϕ_1 is analytic on D and $\|\phi_1\| = 1$.

Suppose $u1$ has a zero of order $n \ge 1$ at $\xi_0 \in D$, i.e., $\lim_{\xi\to\xi_0}[(u1)(\xi)/(\xi - \xi_0)^n]$ exits. So by (3),

$$[u(z)(\xi)/(\xi - \xi_0)^{n-1}]^n = [u1(\xi)/(\xi - \xi_0)^n]^{n-1}u(z^n)(\xi).$$

But $u(z^n)(\xi_0) = 0$ since, by (2),

$$0 = (u1)(\xi_0)u(z^{2n})(\xi_0) = [(uz^n)(\xi_0)]^2.$$

Thus $\lim_{\xi\to\xi_0}[u(z)(\xi)/(\xi - \xi_0)^{n-1}] = 0$, and uz has a zero of order at least n at ξ_0, so ϕ_1 is analytic on D.

To get $\|\phi_1\| \le 1$ note that, by (3), $(uz^n) = (u1)(\xi)\phi_1^n(\xi)$. Thus, if $u1(\xi) \ne 0$, then

$$|\phi_1(\xi)|^n \le \|uz^n\|/|u1(\xi)| \le 1/|u1(\xi)| \quad \text{for all } n.$$

Thus, $|\phi_1(\xi)| \le 1$. Since the zeros of the $u1$ are isolated, we get $\|\phi_1\| \le 1$. That $\|\phi_1\| = 1$ will be noted later.

Finally, putting $f = z$ in (1) we get, for $q \in Q$, $(uz)(q) = (u1)(q)\phi(q)$. Thus $\phi = \phi_1$ on Q. This implies $\|\phi_1\| = 1$. It also implies from (1) that, for $f \in A$, $uf = (u1)f(\phi_1)$ holds on Q. Again since these functions are in H^∞, $uf = (u1)f(\phi_1)$ holds on D, so u is a Type 1 operator.

This finishes the case where $m(Q) > 0$. Now assume $m(Q) = 0$, and we will show that u is a Type 2 isometry. This falls immediately out of Proposition 3. Let $Q, \rho\colon Q \to T$ and $\phi\colon Q \to T$ be as given in Proposition 3. Then

$$\rho(q)u(f)(q) = f(\phi(q)) \quad \text{for } f \in A \text{ and } q \in Q. \tag{4}$$

Let $\psi = \rho^{-1}$. We must show that $uf = E(\psi \cdot f(\phi))$, where $E\colon Y \to A$ is an extension operator on $Y = \{\psi \cdot f(\phi)\colon f \in A\}$. We simply define the operator E on Y by $E(\psi \cdot f(\phi)) = uf$. Then (4) shows that E is an extension operator, and this finishes the proof of Theorem A.

PROOF OF THEOREM B. The construction of $Q \subset T$, $\phi\colon Q \to K$, and $\psi \in C(Q)$ is the same argument as Proposition 3. It only remains to show that $m(Q) = 0$. We suppose $m(Q) > 0$ and get a contradiction. First we show that if F is a closed subset of Q with $m(F) > 0$ then $\phi(F) = K$. For if $k_0 \in K$ and $k_0 \notin \phi(F)$, we can choose $f \in C(K)$ such that $f = 1$ on $\phi(F)$ and $f(k_0) = 0$. Then $u1$ and uf are identical on F, which is impossible since $m(F) > 0$.

Now fix $k_0 \in K$. Then $m(\phi^{-1}(k_0)) = 0$. So there is an open subset U of T with $\phi^{-1}(k_0) \subset U$ and $m(U) < m(Q)/2$. But, letting $F = Q\backslash U$, $m(F) > 0$, and $\phi(F) \neq K$ since $k_0 \notin \phi(F)$. This contradiction shows that $m(Q) = 0$ and Theorem B is proved.

3. Some further remarks. We remarked earlier that one reason for additional isometries on A, besides the natural Type 1 isometries, is that the disc algebra contains subspaces isometric to $C(T)$ which naturally contains A. However, not all Type 2 isometries are restrictions to A of isometries of $C(T)$, as illustrated by the following example (which appears in Rochberg [**31**]).

EXAMPLE. Let Q be a closed subset of T which is homeomorphic to the Cantor set and has Lebesgue measure 0. Let ϕ be a continuous map of Q onto T. By Rudin's Theorem [**25**] there are functions ϕ_1 and ϕ_2 in A such that $\phi_1 = \phi_2 = \phi$ on Q, but $\phi_1 \neq \phi_2$ and $\|\phi_1\| = \|\phi_2\| = 1$. Now define $u\colon A \to A$ by $uf = \frac{1}{2}(f \circ \phi_1 + f \circ \phi_2)$.

It is not difficult to check that u is an isometry which is not Type 1, so it must be Type 2. But there can be no isometry $w\colon C(T) \to A$ such that $w = u$ on A, since then $w(1) = u(1) = 1$, and this is impossible (the argument here is exactly that used in the proof of Example 9.1 in Michael and Pelczynski [**18**]).

We close with a related problem about isometries which seems to be open.

Problem. Describe the isometries of H^∞ into itself.

REFERENCES

1. D. E. Aslpach, *Small into isomorphisms on L_p spaces*, Illinois J. Math. **27** (1983), 300–314.
2. J. Arazy, *The isometries of C_p*, Israel J. Math. **22** (1975), 247–256.
3. Y. Benyamini, *Small into isomorphisms between spaces of continuous functions*, Proc. Amer. Math. Soc. **83** (1981), 479–485.
4. E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc. **25** (1958), 283–290.
5. M. Cambern, *Isometries of certain Banach algebras*, Studia Math. **25** (1965), 217–225.

6. L. Carleson, *Representations of continuous functions*, Math. Z. **66** (1957), 447–451.
7. N. Dunford and J. T. Schwartz, *Linear operators.* I, Interscience, New York, 1958.
8. J. A. Erdos, *A simple proof of Arazy's theorem*, unpublished manuscript.
9. F. Forelli, *The isometries of H^p*, Canad. J. Math. **16** (1964), 721–728.
10. ——, *A theorem on isometries and the application of it to the isometries of $H^p(S)$ for $2 < p < \infty$*, Canad. J. Math. **25** (1973), 284–289.
11. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Engelwood Cliffs, N.J., 1962.
12. W. Holsztynski, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math. **26** (1966), 133–136.
13. C. Kolaski, *Isometries of Bergman spaces on bounded Runge domains*, Canad. J. Math. **33** (1981), 1157–1164.
14. ——, *Isometries of weighted Bergman spaces*, Canad. J. Math. **33** (1981), 1156–1163.
15. A. Koranyi and S. Vagi, *Isometries of H^p spaces of bounded symmetric domains*, Canad. J. Math. **28** (1976), 334–340.
16. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.
17. K. de Leeuw, W. Rudin and J. Wermer, *The isometries of some function spaces*, Proc. Amer. Math. Soc. **11** (1960), 694–698.
18. E. Michael and A. Pelczynski, *A linear extension theorem*, Illinois J. Math. **11** (1967), 563–579.
19. V. D. Pathak, *Isometries of $C^{(n)}[0,1]$*, Pacific J. Math. **94** (1981), 211–222.
20. ——, *Linear isometries of spaces of absolutely continuous functions*, Canad. J. Math. **34** (1982), 298–306.
21. A. Pelczynski, *On simultaneous extension of continuous functions*, Studia Math. **24** (1964), 286–304.
22. ——, *Addendum to: On simultaneous extensions of continuous functions*, Studia Math. **25** (1964), 157–161.
23. ——, *On $C(S)$-subspaces of separable Banach spaces*, Studia Math. **31** (1968), 513–522.
24. N. V. Rao and A. K. Roy, *Linear isometries of some function spaces*, Pacific J. Math. **38** (1971), 177–192.
25. W. Rudin, *Boundary values of continuous analytic functions*, Proc. Amer. Math. Soc. **7** (1956), 808–811.
26. ——, *Real and complex analysis*, McGraw-Hill, New York, 1966.
27. ——, *L^p-isometries and equimeasurability*, Indiana J. Math. **25** (1976), 215–228.
28. R. Schneider, *Isometries of $H^p(U)$*, Canad. J. Math. **25** (1973), 92–95.
29. K. W. Tam, *Isometries of certain function spaces*, Pacific J. Math. **31** (1969), 233–246.
30. F. T. Birtel, editor, *Function algebras* (Proc. Internat. Sympos. Function Algebras, Tulane Univ., 1965), Scott, Foresman, Glenview, Ill., 1966.
31. R. Rochberg, *Which linear maps of the disc algebra are multiplicative*?, Pacific J. Math. **38** (1971), 207–212.
32. J. N. McDonald, *Isometries of function algebras*, Illinois J. Math. **17** (1973), 579–583.
33. ——, *Isometries of the disc algebra*, Pacific J. Math. **58** (1975), 143–154.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74076

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KANSAS NEWMAN COLLEGE, WICHITA, KANSAS 67213