A NOTE ON A WEIGHTED SOBOLEV INEQUALITY

FILIPPO CHIARENZA AND MICHELE FRASCA

ABSTRACT. We give a simple proof of a weighted imbedding theorem whose proof was originally given in [3].

The purpose of this note is to provide a simplified proof of a weighted imbedding theorem previously proved by Fabes, Kenig and Serapioni [3]. They proved the following inequality (see Theorem (1.2) in [3]),

\[ \left( \frac{1}{w(B_R)} \int_{B_R} |u(x)|^{kp} w(x) \, dx \right)^{1/kp} \leq c R \left( \frac{1}{w(B_R)} \int_{B_R} |\nabla u|^p w(x) \, dx \right)^{1/p}. \]

Here \( w \in A_p (1 < p < \infty) \), \( 1 \leq k \leq n/(n - 1) + \delta \) and \( u \) is any function in \( C_0^\infty(B_R) \) (henceforth \( w(B_R) = \int_{B_R} w \, dx \) and \( n \) is the dimension).

Our short proof, inspired by an idea in [4], is an easy consequence of the boundedness of the Hardy-Littlewood maximal operator \( Mf \) between weighted \( L^p \) spaces iff the weight is \( A_p \) (see [5, 2] and, for an elementary proof, the recent paper [1]).

In the following assume the reader is familiar with the relevant definitions and notations as given in [3].

Set \( I_f(x) = \int_{\mathbb{R}^n} |f(y)| |x - y|^{1-n} \, dy \). It is well known that (1) is an immediate consequence of the following inequality:

\[ \left( \frac{1}{w(B_R)} \int_{B_R} |I_f(x)|^{kp} w(x) \, dx \right)^{1/kp} \leq c R \left( \frac{1}{w(B_R)} \int_{B_R} |f(x)|^p w(x) \, dx \right)^{1/p}. \]

Here \( w \in A_p (1 < p < \infty) \), \( 1 \leq k \leq n/(n - 1) + \delta \), \( f \in L^p(B_R, w) \), and \( c, \delta \) are positive constants independent on \( f \) and \( R \).

To prove (2) set, for any \( \varepsilon > 0 \),

\[ I^{(\varepsilon)} f(x) = \int_{|y| \leq \varepsilon} |f(y)| |x - y|^{1-n} \, dy. \]

It is easy to see that \( I^{(\varepsilon)} f(x) \leq c \varepsilon Mf(x) \). Further,

\[ I_f(x) - I^{(\varepsilon)} f(x) = I(\varepsilon) f(x) \]

(3) \[ \leq \|f\|_{L^p(B_R, w)} \left( \int_{\{|x-y|>\varepsilon\} \cap B_R} |x - y|^{1-n} w^{1/(p-1)} \, dy \right)^{1/p} \]

\[ = \left( \frac{1}{p} + \frac{1}{p'} = 1 \right). \]
Because $w \in A_p$, a $q$ can be chosen such that $w \in A_q$, $1 < q < p$, $n - p/q > 0$ (see [2]). Hence,

\[
I(\varepsilon)f(x) \leq c\|f\|_{L^p(B_R, w)} \left( \int_{B_R} [w(y)]^{-1/(q-1)} \, dy \right)^{(q-1)/p} \varepsilon^{1-nq/p}.
\]

(3) and (4) imply

\[
I f(x) \leq c\varepsilon Mf(x) + c\|f\|_{L^p(B_R, w)} \left( \int_{B_R} [w(y)]^{-1/(q-1)} \, dy \right)^{(q-1)/p} \varepsilon^{1-nq/p}.
\]

We minimize with respect to $\varepsilon$ the right side of (5) to get

\[
I f(x) \leq c[Mf(x)]^{1-p/nq} \|f\|_{L^p(B_R, w)}^{p/nq} \left( \int_{B_R} [w(y)]^{-1/(q-1)} \, dy \right)^{(q-1)/nq},
\]

and, using the boundedness of $Mf$ in the $L^p(B_R, w)$ norm,

\[
\|If(x)\|_{L^p(B_R, w)} \leq c\|f\|_{L^p(B_R, w)} \left( \int_{B_R} [w(y)]^{-1/(q-1)} \, dy \right)^{(q-1)/nq},
\]

where $k = nq/(nq - p)$. To complete the proof we divide by $[w(B_R)]^{(nq-p)/npq}$ and use the $A_q$ condition.

\section*{References}


\textsc{Seminario Matematico, Università di Catania, Viale A. Doria, 6, I-95125 Catania, Italy}