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LIE SOLVABLE RINGS

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ABSTRACT. Let $\mathscr{L}(R)$ denote the associated Lie ring of an associative ring R with identity $1 \neq 0$ under the Lie multiplication $[x, y] = xy - yx$ with $x, y \in R$. Further, suppose that the Lie ring $\mathscr{L}(R)$ is solvable of length n. It has been proved that if 3 is invertible in R, then the ideal J of R generated by all elements $[[[x_1, x_2],[x_3, x_4]], x_5]$, $x_1, x_2, x_3, x_4, x_5 \in R$, is nilpotent of index at most $\frac{2}{9}(19 \cdot 10^{n-3} - 1)$ for $n \geqslant 3$. Also, if 2 and 3 are both invertible in R, then the ideal I of R generated by all elements $[x, y]$, $x, y \in R$, is a nil ideal of R. Some applications to Lie solvable group rings are also given.

Let R be any associative ring with identity $1 \neq 0$. We can induce the Lie structure on R by defining the Lie product $[x, y] = xy - yx$ for $x, y \in R$. The Lie ring thus obtained is called the associated Lie ring of R and is denoted by $\mathscr{L}(R)$. Jennings [**1**] proved that if $\mathscr{L}(R)$ is nilpotent then the associative ideal of R generated by all elements $[[x, y], z]$, $x, y, z \in R$, is a nilpotent ideal of R and the ideal generated by all $[x, y]$, $x, y \in R$, is nil. In this paper we study the case when $\mathscr{L}(R)$ is solvable.

1. Lie identities and Lie ideals. Let $x_1, x_2, \ldots, x_n \in R$; then the left normed commutators are defined by $[x_1, x_2] = x_1x_2 - x_2x_1$ and, inductively,

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n].$$

We shall repeatedly use the following well-known identities, which are easy to prove.

LEMMA 1.1 *For $x, y, z \in R$, the following identities are true*:
(i) $[x, y] = -[y, x]$.
(ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (*Jacobi identity*).
(iii) $[xy, z] = x[y, z] + [x, z]y$.
(iv) $[x, yz] = y[x, z] + [x, y]z$.

For any two subsets A and B of R, by $[A, B]$ we shall denote the additive subgroup of R generated by all elements $[a, b]$ with $a \in A$ and $b \in B$. A Lie ideal of R means an ideal of the Lie ring $\mathscr{L}(R)$. Thus, U is a Lie ideal if U is an additive subgroup of R and $[u, r] \in U$ for $u \in U$ and $r \in R$. It is easy to see that $[U, V]$ is a Lie ideal if U and V are Lie ideals.

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Let U be any Lie ideal of R. In view of the identity $ur = [u, r] + ru$, $u \in U$, $r \in R$, the right and left ideals of R, generated by U, are identical. In particular, $RU = UR$ is the two-sided ideal generated by U. Also, U^m consists of finite sums of m-fold products $u_1 u_2 \cdots u_m$ with $u_1, u_2, \ldots, u_m \in U$. Therefore, $(UR)^n = U^n R$ for any Lie ideal U and for any positive integer n.

The derived chain of any Lie ideal U is given by

$$U = \delta^{(0)}(U) \supseteq \delta^{(1)}(U) \supseteq \delta^{(2)}(U) \supseteq \cdots \supseteq \delta^{(n)}(U) \supseteq \cdots,$$

where $\delta^{(n+1)}(U) = [\delta^{(n)}(U), \delta^{(n)}(U)]$, $n \geqslant 0$. We say that $\mathscr{L}(R)$ is solvable of length n if $\delta^{(n)}(\mathscr{L}(R)) = (0)$, n least.

Further, the lower central chain of U is defined by

$$U = \gamma_1(U) \supseteq \gamma_2(U) \supseteq \gamma_3(U) \supseteq \cdots \supseteq \gamma_n(U) \supseteq \cdots,$$

where $\gamma_{n+1}(U) = [\gamma_n(U), U]$, $n \geqslant 1$. The Lie ring $\mathscr{L}(R)$ is nilpotent of class n if $\gamma_{n+1}(\mathscr{L}(R)) = (0)$, n least.

We proceed with a sequence of lemmas needed for our further work. These lemmas are also of independent interest.

LEMMA 1.2. *Let U be a Lie ideal of R; then*
(i) $[U^m, \mathscr{L}(R)] \subseteq [U, \mathscr{L}(R)] \subseteq U$, *and*
(ii) $[\delta^{(1)}(U)R, \delta^{(1)}(U)R] \subseteq U$.

PROOF. (i) follows by induction on m and the identity $[u_1 u_2, r] = [u_1, u_2 r] + [u_2, r u_1]$.

To prove (ii), for $u_1, u_2, u_3, u_4 \in U$ and $r, s \in R$, we have the identity

$$\begin{aligned}[][u_1, u_2]r, [u_3, u_4]s] &= [[u_1, u_2 r], [u_3, u_4 s]] - [u_2[u_1, r], u_4[u_3, s]] \\ &\quad - [u_2[u_1, r], [u_3, u_4]s] + [u_4[u_3, s], [u_1, u_2]r].\end{aligned}$$

This can easily be obtained by expanding the first term on the right and using Lemma 1.1.

Now the right side belongs to U by (i). The lemma follows easily.

LEMMA 1.3. *Let U be a Lie ideal of R; then*
(i) $(\delta^{(1)}(U))^2 \cdot \delta^{(1)}(\mathscr{L}(R)) \subseteq U + [\delta^{(1)}(U), \mathscr{L}(R)]R$, *and*
(ii) $(\delta^{(1)}(U))^4 \cdot \delta^{(2)}(\mathscr{L}(R)) \subseteq \delta^{(1)}(U) + [\delta^{(1)}(U), \mathscr{L}(R)]R$.

PROOF. Let v_1, v_2 be two-fold commutators in the elements of U and let $r_1, r_2 \in R$. Expanding the first term on the right and using Lemma 1.1, we get the identity

$$v_1 v_2 [r_1, r_2] = [v_1 r_1, v_2 r_2] - [v_1, v_2 r_2] r_1 - v_1 [r_1, v_2] r_2.$$

By Lemma 1.2(ii), $[v_1 r_1, v_2 r_2] \in U$, and the other two terms clearly belong to $[\delta^{(1)}(U), \mathscr{L}(R)]R$. This proves (i).

Now let w_1, w_2, w_3, w_4 be two-fold commutators in the elements of U, and let s_1, s_2 be two-fold commutators in the elements of $\mathscr{L}(R) = R$. Then the following identity can be obtained by using Lemma 1.1 and expanding the first term on the right:

$$\begin{aligned} w_1 w_2 w_3 w_4 [s_1, s_2] &= [w_1 w_2 s_1, w_3 w_4 s_2] - w_1 w_2 w_3 [s_1, w_4] s_2 \\ &\quad - w_1 w_2 [s_1, w_3] w_4 s_2 - w_1 [w_2, w_3 w_4 s_2] s_1 - [w_1, w_3 w_4 s_2] w_2 s_1.\end{aligned}$$

The first term on the right, by (i), easily belongs to $\delta^{(1)}(U) + [\delta^{(1)}(U), \mathscr{L}(R)]R$, and all other terms on the right are clearly in $[\delta^{(1)}(U), \mathscr{L}(R)]R$ because it is a two-sided ideal. Thus we get (ii).

This leads to our next lemma. Let J denote the ideal of R generated by all elements $[[x_1, x_2], [x_3, x_4], x_5]$ with $x_1, \ldots, x_5 \in R$. Clearly,

$$J = \left[\delta^{(1)}(\mathscr{L}(R)), \delta^{(1)}(\mathscr{L}(R)), \mathscr{L}(R)\right] R = \left[\delta^{(2)}(\mathscr{L}(R)), \mathscr{L}(R)\right] R.$$

LEMMA 1.4. *For any Lie ideal U of R, we have*

$$\left(\delta^{(1)}(U)\right)^4 \cdot J \subseteq \left[\delta^{(1)}(U), \mathscr{L}(R)\right] R.$$

PROOF. The left side is a finite sum of elements of the type $a[b, c]r$, with $a \in (\delta^{(1)}(U))^4$, $b \in \delta^{(2)}(\mathscr{L}(R))$, $c \in \mathscr{L}(R)$, and $r \in R$. But

$$a[b, c] = [ab, c] - [a, c]b.$$

By Lemma 1.3(ii), $ab \in \delta^{(1)}(U) + [\delta^{(1)}(U), \mathscr{L}(R)]R$. Hence,

$$[ab, c] \in \left[\delta^{(1)}(U), \mathscr{L}(R)\right] R.$$

Also, by Lemma 1.2(i),

$$[a, c]b \in \left[\left(\delta^{(1)}(U)\right)^4, \mathscr{L}(R)\right] R \subseteq \left[\delta^{(1)}(U), \mathscr{L}(R)\right] R.$$

Thus, $a[b, c]r \in [\delta^{(1)}(U), \mathscr{L}(R)]R$ and the lemma is proved.

LEMMA 1.5. *For any Lie ideal U of R, we have*

$$\delta^{(1)}(U)\left[\delta^{(1)}(U), \mathscr{L}(R)\right] \subseteq \gamma_3(U)R.$$

PROOF. Let $u_1, u_2, u_3, u_4 \in U$ and $r \in R$. Then the following identity gives the result:

$$\begin{aligned}[u_1, u_2][u_3, u_4, r] = {} & [[u_3, u_4], [u_1, u_2 r]] - u_2[[u_3, u_4], [u_1, r]] \\ & - [[u_3, u_4], [u_1, u_2]]r - [u_3, u_4, u_2][u_1, r].\end{aligned}$$

COROLLARY 1.6. *Let U be a Lie ideal of R; then*
(i) $[\delta^{(1)}(U), \mathscr{L}(R)]^2 \subseteq \gamma_3(U)R$, *and*
(ii) $J^2 \subseteq \gamma_3(\delta^{(1)}(\mathscr{L}(R)))R$.

PROOF. (i) follows by Lemma 1.5 since $[\delta^{(1)}(U), \mathscr{L}(R)] \subseteq \delta^{(1)}(U)$.
(ii) follows from (i) if we put $U = \delta^{(1)}(\mathscr{L}(R))$ and observe that

$$J = \left[\delta^{(2)}(\mathscr{L}(R)), \mathscr{L}(R)\right] R = R\left[\delta^{(2)}(\mathscr{L}(R)), \mathscr{L}(R)\right].$$

The next lemma is crucial to our further work. Its proof also requires some computations.

LEMMA 1.7. *Let U be a Lie ideal of a ring R in which 3 is invertible. Then $(\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R$.*

PROOF. It is enough to show that $[u_1, u_2, u_3][u_4, u_5, u_6] \in \delta^{(2)}(U)R$ for all $u_1, \ldots, u_6 \in U$. To do this, we proceed as follows. Let

$$a = [u_1, u_2, u_5][u_6, u_4, u_3] + [u_1, u_2, u_6][u_5, u_4, u_3].$$

Observe that the second term can be obtained from the first by interchanging u_5 (the last entry of the first bracket) and u_6 (the first entry of the second bracket). Expanding $[[u_4, u_6u_5, u_3], [u_1, u_2]]$ properly, we can easily get

$$\begin{aligned} a = {} & [[u_4, u_6u_5, u_3], [u_1, u_2]] + [[u_1, u_2, u_5], [u_6, u_4, u_3]] \\ & - u_6[[u_4, u_5, u_3], [u_1, u_2]] - [[u_4, u_6, u_3], [u_1, u_2]]u_5 \\ & - [u_6, u_3][[u_4, u_5], [u_1, u_2]] - [[u_6, u_3], [u_1, u_2]][u_4, u_5] \\ & - [u_4, u_6][[u_5, u_3], [u_1, u_2]] - [[u_4, u_6], [u_1, u_2]][u_5, u_3]. \end{aligned}$$

Certainly, $a \in \delta^{(2)}(U)R$.

In an exactly similar manner,

$$b = [u_1, u_2, u_4][u_6, u_5, u_3] + [u_1, u_2, u_6][u_4, u_5, u_3] \in \delta^{(2)}(U)R.$$

We now turn to the case when the last entry of the first bracket and the last entry of the second bracket are interchanged.

Expanding $[[u_6, u_4], [[u_1, u_2], u_3u_5]]$ and rearranging terms, we get

$$\begin{aligned} c &= [u_1, u_2, u_3][u_6, u_4, u_5] + [u_1, u_2, u_5][u_6, u_4, u_3] \\ &= [[u_6, u_4], [[u_1, u_2], u_3u_5]] + [[u_1, u_2, u_5], [u_6, u_4, u_3]] \\ &\quad - u_3[[u_6, u_4], [u_1, u_2, u_5]] - [[u_6, u_4], [u_1, u_2, u_3]]u_5. \end{aligned}$$

This shows that $c \in \delta^{(2)}(U)R$.

Arguments, as in the case of c, will also give

$$d = [u_1, u_2, u_3][u_6, u_5, u_4] + [u_1, u_2, u_4][u_6, u_5, u_3] \in \delta^{(2)}(U)R$$

and

$$e = [u_1, u_2, u_3][u_4, u_5, u_6] + [u_1, u_2, u_6][u_4, u_5, u_3] \in \delta^{(2)}(U)R.$$

Finally, by using Lemma 1.1(ii) and rearranging terms, we get

$$3[u_1, u_2, u_3][u_4, u_5, u_6] = a - b - c + d + 2e \in \delta^{(2)}(U)R.$$

Since 3 is invertible in R,

$$[u_1, u_2, u_3][u_4, u_5, u_6] \in \delta^{(2)}(U)R.$$

This completes the proof.

COROLLARY 1.8. *If* 3 *is invertible in* R, *then* $J^4 \subseteq \delta^{(3)}(\mathscr{L}(R))R$.

PROOF. $J^2 \subseteq \gamma_3(\delta^{(1)}(\mathscr{L}(R)))R$ by Corollary 1.6. Therefore,

$$\begin{aligned} J^4 &\subseteq \left(\gamma_3(\delta^{(1)}(\mathscr{L}(R)))R\right)^2 = \left(\gamma_3(\delta^{(1)}(\mathscr{L}(R)))\right)^2 R \\ &\subseteq \delta^{(2)}\left(\delta^{(1)}(\mathscr{L}(R))\right)R \quad \text{(by Lemma 1.7)} \\ &= \delta^{(3)}(\mathscr{L}(R))R. \end{aligned}$$

The next lemma does not assume that 3 is invertible in R and has a much simpler proof than Lemma 1.7.

LEMMA 1.9. *Let* U *be a Lie ideal of* R *such that* U *is also a subring of* R; *then* $(\gamma_3(U))^2 \subseteq \delta^{(2)}(U)R$.

PROOF. Expanding the first term on the right, we have

$$\begin{aligned}[u_1, u_2, u_3][u_4, u_5, u_6] = {} & [[u_4, u_5], [[u_1, u_2]u_6, u_3]] \\ & + [[u_1, u_2, u_3], [u_4, u_5]]u_6 + [u_1, u_2][[u_6, u_3], [u_4, u_5]] \\ & + [[u_1, u_2], [u_4, u_5]][u_6, u_3]\end{aligned}$$

for all $u_1, \dots, u_6 \in U$. Clearly the right side belongs to $\delta^{(2)}(U)R$, as desired.

COROLLARY 1.10. *For any ring R, $(\gamma_3(\mathscr{L}(R)))^2 \subseteq \delta^{(2)}(\mathscr{L}(R))R$.*

2. Main results. In this section we prove our main theorems.

THEOREM 2.1. *Let R be a ring in which 3 is invertible, and let its associated Lie ring $\mathscr{L}(R)$ be solvable of length $n \geqslant 3$. Then the ideal J of R, generated by all elements $[[x_1, x_2], [x_3, x_4], x_5]$ with $x_1, \dots, x_5$ in R, is nilpotent of index at most $\frac{2}{9}(19 \cdot 10^{n-3} - 1)$.*

PROOF. If $\mathscr{L}(R)$ is solvable of length $n = 3$, then $\delta^{(3)}(\mathscr{L}(R)) = (0)$. By Corollary 1.8, $J^4 = (0)$. So the theorem is true for $n = 3$.

We assume that $n \geqslant 4$. Now for any Lie ideal U of R, using Lemmas 1.4 and 1.5, we get

$$\delta^{(1)}(U)\left(\delta^{(1)}(U)\right)^4 J \subseteq \gamma_3(U)R.$$

Thus, by Lemma 1.7,

$$\left\{\left(\delta^{(1)}(U)\right)^5 J\right\}^2 \subseteq \left(\gamma_3(U)R\right)^2 = \left(\gamma_3(U)\right)^2 R \subseteq \delta^{(2)}(U)R.$$

Putting $U = \delta^{(m-2)}(\mathscr{L}(R))$, we get

$$\left\{\left(\delta^{(m-1)}(\mathscr{L}(R))\right)^5 J\right\}^2 \subseteq \delta^{(m)}(\mathscr{L}(R))R \quad \text{for all } m \geqslant 4.$$

Thus, for $m = 4$, we have

$$\left\{\left(\delta^{(3)}(\mathscr{L}(R))\right)^5 J\right\}^2 \subseteq \delta^{(4)}(\mathscr{L}(R))R$$

and, using Corollary 1.8,

$$\left\{(J^4)^5 J\right\}^2 = J^{(1+2\cdot 10)2} \subseteq \delta^{(4)}(\mathscr{L}(R))R.$$

We claim that, by induction on m,

$$J^{2\{1+10+10^2+\cdots+10^{m-4}+2\cdot 10^{m-3}\}} \subseteq \delta^{(m)}(\mathscr{L}(R))R \quad \text{for all } m \geqslant 4.$$

Assume this is true for m and use $\{(\delta^{(m)}(\mathscr{L}(R)))^5 J\}^2 \subseteq \delta^{(m+1)}(\mathscr{L}(R))R$ to prove it for $m + 1$. Thus,

$$J^N \subseteq \delta^{(n)}(\mathscr{L}(R))R = (0),$$

where

$$N = 2\{1 + 10 + 10^2 + \cdots + 10^{n-4} + 2 \cdot 10^{n-3}\} = \tfrac{2}{9}[19 \cdot 10^{n-3} - 1],$$

as desired.

Next, we prove that the ideal I of R, generated by all elements $[x, y]$, $x, y \in R$, is a nil ideal if 2 and 3 are both invertible in R. First, we need the following

LEMMA 2.2. *Let R be a ring in which 2 is invertible. Then*
(i) $[[x_1, x_2], [x_3, x_4]]^3 \in J$ *for all* $x_1, \ldots, x_4 \in R$,
(ii) $[x, y, z]^{10} \in J$ *for all* $x, y, z \in R$, *and*
(iii) $[x, y]^{21} \in J$ *for all* $x, y \in R$.

PROOF. Expanding the first term on the right, we get

$$\begin{aligned}2[[x_1, x_2], [x_3, x_4]]^2 = {} & \big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]\big] \\ & - [x_1, x_2][[x_1, x_2], [x_3, x_4], [x_3, x_4]] \\ & - [[x_1, x_2], [x_3, x_4], [x_3, x_4]][x_1, x_2].\end{aligned}$$

Thus,

$$2[[x_1, x_2], [x_3, x_4]]^2 \equiv \big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]\big] \pmod{J}.$$

Also,

$$\begin{aligned}&2\big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]\big][[x_1, x_2], [x_3, x_4]] \\ &\quad = \big[[x_3, x_4]^2, [x_1, x_2]^2, [x_3, x_4], [x_1, x_2]\big] \\ &\quad\quad + \big[\big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4]\big], [[x_1, x_2], [x_3, x_4]]\big] \\ &\quad\quad + \big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4], [x_1, x_2]\big][x_3, x_4] \\ &\quad\quad + [x_3, x_4]\big[[x_1, x_2]^2, [x_3, x_4], [x_3, x_4], [x_1, x_2]\big] \\ &\quad \equiv 0 \pmod{J}.\end{aligned}$$

Combining, we get $4[[x_1, x_2], [x_3, x_4]]^3 \in J$, and, since 2 is invertible, we get (i).

To prove (ii), observe the identity

$$[x, y, z]^2 = [[x, y], [[x, y]z, z]] + [[x, y, z], [x, y]]z,$$

and use (i), keeping in view that

$$r[[x_1, x_2], [x_3, x_4]] = [[x_1, x_2], [x_3, x_4]]r - [[x_1, x_2], [x_3, x_4], r].$$

Similarly, to prove (iii) it is enough to use (i) and see that

$$[x, y]^3 = [[xy, y], [yx, x]] + [[y, x], [xy, y]]x + [[yx, x], [x, y]]y.$$

Note that powers given in Lemma 2.2 are not the best possible; the purpose is served by proving that some power in each case belongs to J.

THEOREM 2.3. *Let R be a ring in which both 2 and 3 are invertible, and let I_0 be the ideal of R generated by all elements $[x, y, z]$, $x, y, z \in R$. If the associated Lie ring $\mathscr{L}(R)$ is solvable, then I_0 is a nil ideal.*

PROOF. Clearly, $I_0 = \gamma_3(\mathscr{L}(R))R = R\gamma_3(\mathscr{L}(R))$. By Corollary 1.10, $I_0^2 = (\gamma_3(\mathscr{L}(R)))^2R \subseteq \delta^{(2)}(\mathscr{L}(R))R$. Now suppose $\mathscr{L}(R)$ is solvable of length n, so $\delta^{(n)}(\mathscr{L}(R)) = (0)$. If $n = 1$, $I_0 = (0)$. If $n = 2$, $I_0^2 = (0)$. So assume $n \geqslant 3$.

Let $\alpha \in I_0$; then $\alpha^2 \in I_0^2 \subseteq \delta^{(2)}(\mathscr{L}(R))R$ and, hence,

$$\alpha^2 = \sum_{i=1}^{m} [[x_i, y_i], [u_i, v_i]] r_i = \sum_{i=1}^{m} \alpha_i r_i,$$

where $\alpha_i = [[x_i, y_i], [u_i, v_i]]$ for $i = 1, 2, \ldots, m$. By Lemma 2.2(i) each $\alpha_i^3 \in J$. Also, $[\alpha_i, r] \in J$ for every $r \in R$. Further, $\alpha_i \beta \alpha_i = \alpha_i^2 \beta - \alpha_i[\alpha_i, \beta]$ for any $\beta \in R$. Thus,

$$\alpha_i \beta \alpha_i \equiv \alpha_i^2 \beta \pmod J$$

and, similarly,

$$\alpha_i \beta \alpha_i \theta \alpha_i \equiv \alpha_i^3 \beta \theta \,(\mathrm{mod}\, J) \equiv 0 \,(\mathrm{mod}\, J)$$

as $\alpha_i^3 \in J$.

The above arguments, applied to $(\alpha^2)^k = (\sum_1^m \alpha_i r_i)^k$ for $k \geqslant 2m + 1$, immediately give that $(\alpha^2)^k \in J$ for $k \geqslant 2m + 1$. But, by Theorem 2.1, J is nilpotent, hence $((\alpha^2)^k)^N = 0$ for suitable N. This proves that α is nilpotent for every $\alpha \in I_0$. That is, I_0 is a nil ideal.

In fact, we are able to obtain a much stronger result.

THEOREM 2.4. *Let R be a ring in which both* 2 *and* 3 *are invertible, and let I be the ideal of R generated by all elements $[x, y]$, $x, y \in R$. If the associated Lie ring $\mathscr{L}(R)$ is solvable, then I is a nil ideal.*

PROOF. $I = \gamma_2(\mathscr{L}(R))R = \delta^{(1)}(\mathscr{L}(R))R$. Let $\mathscr{L}(R)$ be solvable of length n; then $\delta^{(n)}(\mathscr{L}(R)) = (0)$. Therefore, if $n = 1$, $I = (0)$.

Let $\alpha \in I$; then

$$\alpha = \sum_{i=1}^{m} [x_i, y_i] r_i = \sum_{i=1}^{m} \alpha_i r_i,$$

where $\alpha_i = [x_i, y_i]$. By Lemma 2.2(iii), $\alpha_i^{21} = [x_i, y_i]^{21}$ always belongs to J.

Now,

$$[x, y]r[x, y] = [x, y]^2 r - [x, y][x, y, r] \equiv [x, y]^2 r \pmod{I_0}.$$

If we take $\alpha^k = (\sum_{i=1}^m \alpha_i r_i)^k$ for $k \geqslant 20m + 1$, then α^k will be a finite sum of k-fold products of elements from $\{\alpha_1 r_1, \alpha_2 r_2, \ldots, \alpha_m r_m\}$, and in each k-fold product some α_i will be repeated at least 21 times. Collecting repeatedly these factors, by the above process, the k-fold products will be congruent to $\alpha_i^{21} r \pmod{I_0}$. Since $\alpha_i^{21} \in J$, this implies that $\alpha^k = \lambda + \mu$ with $\lambda \in J$ and $\mu \in I_0$. But, by Theorem 2.3, I_0 is a nil ideal, so $\mu^l = 0$ for some l. This gives $(\alpha^k)^l = \alpha^{kl} \in J$. Now use the nilpotency of J to get $(\alpha^{kl})^N = 0$. Thus, α is nilpotent and I is a nil ideal.

3. Applications to group rings. Lie solvable group rings were studied by Passi, Passman and Sehgal in [**2**]. Let $K[G]$ denote the group ring of the group G over the field K with Char $K = p \geqslant 0$, $p \neq 2$. If $p > 0$ we say that a group G is p-Abelian if the commutator subgroup G' is a finite p-group. For convenience, 0-Abelian will mean Abelian. It was proved in [**2**] that if Char $K = p \neq 2$, then the associated Lie algebra $\mathscr{L}(K[G])$ of $K[G]$ is solvable if and only if G is p-Abelian. Using Theorem 2.4 we get an alternative proof in characteristic 0 as follows.

Suppose $\operatorname{Char} K = 0$ and $\mathscr{L}(K[G])$ is solvable. Then, by Theorem 2.4, $I = \gamma_2(\mathscr{L}(K[G])) \cdot K[G]$ is a nil ideal of $K[G]$. So, by [**3**, Theorem 2.3.4, p. 47], $I = (0)$. This gives $\gamma_2(\mathscr{L}(K[G])) = (0)$, i.e., $K[G]$ is commutative. Hence, G is Abelian. Thus, $\mathscr{L}(K[G])$ is solvable if and only if G is Abelian.

Also in $\operatorname{Char} K = p > 0$, $p \neq 2, 3$, we have an advantage. Suppose $\mathscr{L}(K[G])$ is solvable. Then $I = [K[G], K[G]]K[G] = \omega(K[G']) \cdot K[G]$ is a nil ideal of $K[G]$ by Theorem 2.4. Hence, $\omega(K[G']) \cdot K[G]$ is contained in the Jacobson radical $J(K[G])$ of $K[G]$. By [**3**, Lemma 10.1.13, p. 415], G' is a p-group. Also, $\mathscr{L}(K[G])$ is solvable implies $K[G]$ satisfies a polynomial identity. By [**3**, Theorem 5.2.14, p. 189], $|G : \Delta(G)| < \infty$ and $\Delta'(G)$ is finite, where $\Delta(G)$ is the FC-subgroup of G. Now $G'/G' \cap \Delta(G)$ is a finite p-group, $G' \cap \Delta(G)/\Delta'(G)$ is Abelian, and $\Delta'(G)$ is a finite p-group implies G', hence G, is solvable. Thus, G' is a locally finite p-group, since it is a solvable p-group. Also, as in [**2**], if it is proved that G is an FC-group, then G is p-Abelian by the above argument. Perhaps our results lead to a different motivation.

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