BOUNDNESS OF THE MAXIMAL OPERATOR
ON WEIGHTED BMO
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ABSTRACT. The Hardy-Littlewood maximal operator $M^*$ is a bounded op-
erator mapping $\text{BMO}_w$ into $\text{BLO}_w$ if and only if the weight $w$ is a Reverse
Hölder weight in weak $\alpha_2$.

$T$ will denote the unit circle in the complex plane. For an interval $I \subset T$, $I(f) = (1/|I|) \int_I f(x) \, dx$. A weight $w \in \text{RH}$ if $w$ satisfies the Reverse Hölder Inequality:
There exists a $p > 1$ and a constant $C$ such that $I(w^p)^{1/p} \leq CI(w)$ for all intervals $I$. $f$ is a function of weighted bounded mean oscillation, $f \in \text{BMO}_w$, provided
$$I(|f - I(f)|) \leq CT(w) \quad \text{for all intervals } I,$$
while $f$ is a function of weighted bounded lower oscillation, $f \in \text{BLO}_w$, if
$$I(f) - \text{ess inf}_I f \leq CI(w) \quad \text{for all } I.$$

A weight $w$ belongs to the Kerman-Torchinsky class $\alpha_p$ if for any interval $I$ and measurable set $E \subset I$ [5],
$$\left( \frac{|E|}{|I|} \right)^p \leq C \frac{w(E)}{w(I)}.$$
$\alpha_p \subset \text{RH}$. Indeed, $A_p \subset \alpha_p \subset \bigcap_{q>p} A_q$, where $A_p$ is the Muckenhoupt class of weights. In [2], we introduced the class weak $\alpha_p$, defined by (1) with the additional proviso that $E$ be an interval. This class properly contains $\alpha_p$. For if $w$ is a doubling measure with constant $2^p$, that is
$$w(J) \leq 2^pw(I) \quad \text{for all intervals } I \subset J$$
with $|J| = 2|I|$, then a standard argument shows $w \in \text{weak } \alpha_p$. But there exist doubling measures that are not in $\text{RH}$ [4]. It is not yet known whether $\alpha_p$ and $\text{RH} \cap \text{weak } \alpha_p$ are the same.

Let $M^*$ denote the Hardy-Littlewood maximal operator
$$M^*f(x) = \sup_{I \ni x} I(|f|).$$
We will show the following result.

THEOREM. $M^*: \text{BMO}_w \to \text{BLO}_w$ is a bounded operator if and only if $w \in \text{RH} \cap \text{weak } \alpha_2$.

The necessity of the condition $\text{RH} \cap \text{weak } \alpha_2$ was proven in [2], but a stronger condition was used for the converse. We prove the sufficiency here:
Fix an interval $I$, and assume, with no loss of generality, that $f \geq 0$, $f \in \text{BMO}_w$. For $x \in I$, define

$$F_1(x) = \sup\{J(f) : x \in J, J \subseteq 4I\},$$

$$F_2(x) = \sup\{J(f) : x \in J, J \not\subseteq 4I\},$$

where $4I$ is the interval concentric with $I$ but of four times the length. So on $I$, $M^*f(x) = \max\{F_1(x), F_2(x)\}$. Put

$$E_1 = \{x \in I : F_1(x) \geq F_2(x)\},$$

$$E_2 = \{x \in I : F_2(x) > F_1(x)\}$$

and $\beta = \text{ess inf}_I M^*f$.

It will suffice to show

$$\frac{1}{|I|} \int_{E_i} (F_i - \beta) \leq CI(w) \quad \text{for } i = 1 \text{ and } 2.$$

For $i = 1$, following the proof in [2], we obtain disjoint intervals $I_n \subset 4I$, intervals $J_n \supset I_n$ of twice their length, and a “bad” function

$$b = \sum_{n} (f - J_n(f)) \chi_{I_n}.$$

The proof of (2) requires showing $b \in L^r$ for some $r > 1$ with

$$\|b\|_r \leq C \left( \int_{4I} w^r \right)^{1/r}.$$

As $w \in \text{RH}$, there exists an $r > 1$ and $p > 1$ such that $w^r \in A_p$, $w^{-rq/p} \in A_q$, where $1/p + 1/q = 1$, and $J(w^r) \leq C J(w)^r$ for all intervals $J$ [3]. Now

$$\|b\|_r^r = \sum_n \int_{J_n} |f - J_n(f)|^r w^{-rq/p} w^{r/p} \leq \sum_n \left( \int_{J_n} |f - J_n(f)|^{qr} w^{-qr/p} \right)^{1/q} \left( \int_{J_n} w^r \right)^{1/p}. $$

Let $f_n^*(x) = \sup\{J|f - J(f)| : x \in J \subseteq J_n\}$ and $w_n^*(x) = \sup\{J(w) : x \in J \subseteq J_n\}$; that is, the sharp and maximal functions restricted to $J_n$. By the weighted version of the Sharp Function Theorem [1],

$$\left( \int_{J_n} |f - J_n(f)|^{qr} w^{-qr/p} \right)^{1/q} \leq C \left( \int_{J_n} (f_n^*)^{qr} w^{-qr/p} \right)^{1/q}.$$

But with $f \in \text{BMO}_w$, $f_n^*(x) \leq C w_n^*(x) \leq C [((w^r)_n^*)^{-1}]^{1/r}$, using Hölder’s Inequality. This, Muckenhoupt’s Theorem, and the fact that $w^r$ is a doubling measure give

$$\|b\|_r \leq C \sum_n \left( \int_{J_n} [(w^r)_n^*]^{q} w^{-qr/p} \right)^{1/q} \left( \int_{J_n} w^r \right)^{1/p} \leq C \sum_n \int_{J_n} w^r \leq C \int_{4I} w^r$$

which is (3).

For $i = 2$, we actually show

$$F_2(x) - \beta \leq CI(w) \quad \text{for all } x \in E_2.$$
which gives (2). We analyze the case where \(x \in E_2 \cap J, J \not\subseteq 4I, I = [a, a + h]\) and \(J = [b, c]\) with \(b \in I\). Set \(\tilde{J} = [b - h, c]\), \(J_0 = [b - h, b]\) and \(J_k = [b, b + 2^{k-1}h]\) as long as \(2^{k-1}h < c - b\). Let \(n\) be the smallest integer with \(2^{n-1}h \geq c - b\), and call \(J = J_n\). Since \(I \subseteq \tilde{J}\), \(\tilde{J}(f) \leq \beta\). Hence,

\[
J(f) - \beta \leq J(f) - \tilde{J}(f) = \left( \frac{1}{|J|} - \frac{1}{|\tilde{J}|} \right) \int_J f - \frac{1}{|J|} \int_{J_0} f
\]

\[
= \frac{h}{|J|} [J(f) - J_0(f)]
\]

\[
\leq 2^{2-n} [J_1(f) - J_0(f) + J_n(f) - J_1(f)].
\]

Now let \(R = J_0 \cup J_1\). Then

\[
|J_1(f) - J_0(f)| \leq |J_1(f) - R(f)| + |J_0(f) - R(f)| \leq \frac{4}{|R|} \int_R |f - R(f)|
\]

\[
\leq CR(w) \quad \text{since } f \in \text{BMO}_w.
\]

Also,

\[
|J_n(f) - J_1(f)| = \left| \sum_{k=1}^{n-1} J_{k+1}(f) - J_k(f) \right|
\]

\[
\leq 2 \sum_{k=1}^{n-1} J_{k+1}(|f - J_{k+1}(f)|) \leq C \sum_{k=1}^{n-1} J_{k+1}(w).
\]

By the weak \(\alpha_2\) condition,

\[
J_{k+1}(w) \leq C \frac{|J_{k+1}|}{|J_1|} J_1(w) \leq C 2^k J_1(w).
\]

Hence,

\[
|J_n(f) - J_1(f)| \leq CJ_1(w) \sum_{k=1}^{n-1} 2^k \leq C 2^{n+1} R(w).
\]

Thus

\[
J(f) - \beta \leq 2^{2-n}[C_1R(w) + C_22^{n+1}R(w)] \leq CR(w).
\]

But \(w\) is a doubling measure [3], so \(R(w) \leq CI(w)\) and (4) follows.

REFERENCES