PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 1, May 1985

BOUNDEDNESS OF THE MAXIMAL OPERATOR ON WEIGHTED BMO

STEVEN BLOOM

ABSTRACT. The Hardy-Littlewood maximal operator M^* is a bounded operator mapping BMO_w into BLO_w if and only if the weight w is a Reverse Hölder weight in weak α_2.

$\mathbf{T}$ will denote the unit circle in the complex plane. For an interval $I \subset \mathbf{T}$, $I(f) = (1/|I|)\int_I f(x)\,dx$. A weight $w \in \mathrm{RH}$ if w satisfies the Reverse Hölder Inequality: There exists a $p > 1$ and a constant C such that $I(w^p)^{1/p} \leq CI(w)$ for all intervals I. f is a function of weighted bounded mean oscillation, $f \in \mathrm{BMO}_w$, provided

$$I(|f - I(f)|) \leq CT(w) \quad \text{for all intervals } I,$$

while f is a function of weighted bounded lower oscillation, $f \in \mathrm{BLO}_w$, if

$$I(f) - \operatorname*{ess\,inf}_I f \leq CI(w) \quad \text{for all } I.$$

A weight w belongs to the Kerman-Torchinsky class α_p if for any interval I and measurable set $E \subset I$ [**5**],

$$\left(\frac{|E|}{|I|}\right)^p \leq C\frac{w(E)}{w(I)}. \tag{1}$$

$\alpha_p \subset \mathrm{RH}$. Indeed, $A_p \subset \alpha_p \subset \bigcap_{q>p} A_q$, where A_p is the Muckenhoupt class of weights. In [**2**], we introduced the class weak α_p, defined by (1) with the additional proviso that E be an interval. This class properly contains α_p. For if w is a doubling measure with constant 2^p, that is

$$w(J) \leq 2^p w(I) \quad \text{for all intervals } I \subset J$$

with $|J| = 2|I|$, then a standard argument shows $w \in$ weak α_p. But there exist doubling measures that are not in RH [**4**]. It is not yet known whether α_p and $\mathrm{RH} \cap$ weak α_p are the same.

Let M^* denote the Hardy-Littlewood maximal operator

$$M^*f(x) = \sup_{I \ni x} I(|f|).$$

We will show the following result.

THEOREM. *$M^*\colon \mathrm{BMO}_w \to \mathrm{BLO}_w$ is a bounded operator if and only if $w \in \mathrm{RH} \cap$ weak α_2.*

The necessity of the condition $\mathrm{RH} \cap$ weak α_2 was proven in [**2**], but a stronger condition was used for the converse. We prove the sufficiency here:

Received by the editors June 19, 1984.
1980 *Mathematics Subject Classification*. Primary 42B25; Secondary 42A50.

0002-9939/85 $1.00 + $.25 per page

Fix an interval I, and assume, with no loss of generality, that $f \geq 0$, $f \in \mathrm{BMO}_w$. For $x \in I$, define

$$F_1(x) = \sup\{J(f)\colon x \in J,\ J \subset 4I\}, \quad F_2(x) = \sup\{J(f)\colon x \in J, J \not\subset 4I\},$$

where $4I$ is the interval concentric with I but of four times the length. So on I, $M^* f(x) = \max\{F_1(x), F_2(x)\}$. Put

$$E_1 = \{x \in I\colon F_1(x) \geq F_2(x)\}, \quad E_2 = \{x \in I\colon F_2(x) > F_1(x)\}$$

and $\beta = \operatorname{ess\,inf}_I M^* f$.

It will suffice to show

$$\frac{1}{|I|}\int_{E_i}(F_i - \beta) \leq CI(w) \quad \text{for } i = 1 \text{ and } 2. \tag{2}$$

For $i = 1$, following the proof in [**2**], we obtain disjoint intervals $I_n \subset 4I$, intervals $J_n \supset I_n$ of twice their length, and a "bad" function

$$b = \sum_n (f - J_n(f))\chi_{I_n}.$$

The proof of (2) requires showing $b \in L^r$ for some $r > 1$ with

$$\|b\|_r \leq C\left(\int_{4I} w^r\right)^{1/r}. \tag{3}$$

As $w \in \mathrm{RH}$, there exists an $r > 1$ and $p > 1$ such that $w^r \in A_p$, $w^{-rq/p} \in A_q$, where $1/p + 1/q = 1$, and $J(w^r) \leq CJ(w)^r$ for all intervals J [**3**]. Now

$$\begin{aligned}\|b\|_r^r &= \sum_n \int_{I_n} |f - J_n(f)|^r w^{-r/p} w^{r/p} \\ &\leq \sum_n \left(\int_{J_n} |f - J_n(f)|^{qr} w^{-qr/p}\right)^{1/q} \left(\int_{J_n} w^r\right)^{1/p}.\end{aligned}$$

Let $f_n^\sharp(x) = \sup\{J|f - J(f)|\colon x \in J \subset J_n\}$ and $w_n^*(x) = \sup\{J(w)\colon x \in J \subset J_n\}$; that is, the sharp and maximal functions restricted to J_n. By the weighted version of the Sharp Function Theorem [**1**],

$$\left(\int_{J_n} |f - J_n(f)|^{qr} w^{-qr/p}\right)^{1/q} \leq C\left(\int_{J_n} (f_n^\sharp)^{qr} w^{-qr/p}\right)^{1/q}.$$

But with $f \in \mathrm{BMO}_w$, $f_n^\sharp(x) \leq Cw_n^*(x) \leq C[(w^r)_n^*(x)]^{1/r}$, using Hölder's Inequality. This, Muckenhoupt's Theorem, and the fact that w^r is a doubling measure give

$$\begin{aligned}\|b\|_r^r &\leq C\sum_n \left(\int_{J_n} [(w^r)_n^*]^q w^{-qr/p}\right)^{1/q} \left(\int_{J_n} w^r\right)^{1/p} \\ &\leq C\sum_n \int_{J_n} w^r \leq C\sum_n \int_{I_n} w^r \leq C\int_{4I} w^r\end{aligned}$$

which is (3).

For $i = 2$, we actually show

$$F_2(x) - \beta \leq CI(w) \quad \text{for all } x \in E_2 \tag{4}$$

which gives (2). We analyze the case where $x \in E_2 \cap J$, $J \not\subset 4I$, $I = [a, a+h]$ and $J = [b, c]$ with $b \in I$. Set $\overline{J} = [b-h, c]$, $J_0 = [b-h, b]$ and $J_k = [b, b+2^{k-1}h]$ as long as $2^{k-1}h < c - b$. Let n be the smallest integer with $2^{n-1}h \geq c - b$, and call $J = J_n$. Since $I \subset \overline{J}$, $\overline{J}(f) \leq \beta$. Hence,

$$\begin{aligned} J(f) - \beta \leq J(f) - \overline{J}(f) &= \left(\frac{1}{|J|} - \frac{1}{|\overline{J}|}\right)\int_J f - \frac{1}{|\overline{J}|}\int_{J_0} f \\ &= \frac{h}{|\overline{J}|}[J(f) - J_0(f)] \\ &\leq 2^{2-n}[J_1(f) - J_0(f) + J_n(f) - J_1(f)]. \end{aligned}$$

Now let $R = J_0 \cup J_1$. Then

$$\begin{aligned} |J_1(f) - J_0(f)| \leq |J_1(f) - R(f)| + |J_0(f) - R(f)| &\leq \frac{4}{|R|}\int_R |f - R(f)| \\ &\leq CR(w) \quad \text{since } f \in \mathrm{BMO}_w. \end{aligned}$$

Also,

$$\begin{aligned} |J_n(f) - J_1(f)| &= \left|\sum_{k=1}^{n-1} J_{k+1}(f) - J_k(f)\right| \\ &\leq 2\sum_{k=1}^{n-1} J_{k+1}(|f - J_{k+1}(f)|) \leq C\sum_{k=1}^{n-1} J_{k+1}(w). \end{aligned}$$

By the weak α_2 condition,

$$J_{k+1}(w) \leq C\frac{|J_{k+1}|}{|J_1|}J_1(w) \leq C2^k J_1(w).$$

Hence,

$$|J_n(f) - J_1(f)| \leq CJ_1(w)\sum_{k=1}^{n-1} 2^k \leq C2^{n+1}R(w).$$

Thus

$$J(f) - \beta \leq 2^{2-n}[C_1R(w) + C_2 2^{n+1}R(w)] \leq CR(w).$$

But w is a doubling measure [**3**], so $R(w) \leq CI(w)$ and (4) follows.

REFERENCES

1. S. Bloom, *A commutator theorem and weighted* BMO (to appear).
2. ——, *The maximal function on weighted* BMO, Harmonic Analysis (Proc. Conf., Cortona, Italy, 1982), pp. 227–239.
3. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 241–250.
4. C. Fefferman and B. Muckenhoupt, *Two nonequivalent conditions for weight functions,* Proc. Amer. Math. Soc. **45** (1974), 99–104.
5. R. A. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function,* Studia Math. **71** (1981/82), 277–284.

DEPARTMENT OF MATHEMATICS, SIENA COLLEGE, LOUDONVILLE, NEW YORK 12211