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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

M. RAMA MOHANA RAO AND P. SRINIVAS

ABSTRACT. The asymptotic behavior of solutions of Volterra integro-differential equations of the form

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)\,ds + F(t)$$

is discussed in which A is not necessarily a stable matrix. An equivalent equation which involves an arbitrary function is derived and a proper choice of this function would pave a way for the new coefficient matrix B (corresponding A) to be stable.

1. Introduction. The objective of this paper is to investigate the asymptotic behavior of solutions of the Volterra integro-differential equation (VIDE)

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)\,ds + F(t) \tag{1.1}$$

where $A(t)$ and $K(t,s)$ are $n \times n$ matrices defined and continuous on $0 \le t < \infty$ and $0 \le s \le t < \infty$, respectively, and $x(t)$ and $F(t)$ are n-vectors with $F(t)$ continuous on $0 \le t < \infty$, when the matrix A is not necessarily stable. Our main approach here is by way of deriving an equivalence theorem (Lemma 2.1) which has the potential to supply us a stable matrix B corresponding to A.

It is well known that the linear autonomous ordinary differential system is asymptotically stable if all the characteristic roots of the coefficient matrix have negative real parts [**12**, Chapter 3]. For nonautonomous systems, with an addition of the Lipschitz condition on the coefficient matrix, similar results have been expounded in [**4** and **5**]. Thus while studying VIDE (1.1), be it through Liapunov second method [**1**, **2**, **14**, **15**] or from perturbation theory [**8**, **10**, **13**], it has invariably been assumed that the coefficient matrix is stable. Notable exceptions that have dispensed with the stability condition on the coefficient matrix have been the works of Levin [**9**], Grossman and Miller [**7**], Grimmer and Seifert [**6**], Burton [**3**], among others. In [**9**] this has been done by defining a suitable energy function while in [**7**] the integrability of the resolvent function of VIDE (1.1) has been characterized by a transformation condition similar to that given in [**11**] for Volterra integral equations. In [**6**] the same has been achieved by studying the rsolvent of a transformed equation. Quite recently in [**3**], the conditions involving the anti-derivatives of the kernel are assumed. Motivated by the interesting nature of this problem, an attempt has been made in §2 to study the asymptotic behavior of solutions of (1.1)

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when the coefficient matrix A in (1.1) is not necessarily stable. Our approach here has been to develop an equivalent equation which involves an arbitrary function. A proper choice of this function would pave a way for the new coefficient matrix (corresponding to equivalent VIDE) to be stable.

2. Main results. The following lemmas are useful in our subsequent discussion.

LEMMA 2.1. *Let $\Phi(t,s)$ be an $n \times n$ continuously differentiable matrix function on $0 \le s \le t < \infty$. Then the equation (1.1) is equivalent to*

$$y'(t) = B(t)y(t) + \int_0^t L(t,s)y(s)\,ds + H(t), \qquad y(0) = x_0, \tag{2.1}$$

where

$$B(t) = A(t) - \Phi(t,t), \tag{2.2}$$

$$L(t,s) = K(t,s) + \Phi_s(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s)\,du,$$

and

$$H(t) = F(t) + \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F(s)\,ds.$$

PROOF. Let $x(t)$ with $x(0) = x_0$ be any solution of (1.1) existing on the interval $0 \le t < \infty$. Consider the identity

$$\int_0^t \Phi_s(t,s)x(s)\,ds = \Phi(t,t)x(t) - \Phi(t,0)x_0 - \int_0^t \Phi(t,s)x'(s)\,ds.$$

Substituting for $x'(t)$ from (1.1) and using Fubini's theorem, we get

$$\begin{aligned}\int_0^t \Phi_s(t,s)x(s)\,ds = {}& \Phi(t,t)x(t) - \Phi(t,0)x_0 - \int_0^t \Phi(t,s)F(s)\,ds \\ & - \int_0^t \Phi(t,s)A(s)x(s)\,ds - \int_0^t \left[\int_\tau^t \Phi(t,s)K(s,\tau)\,ds\right] x(\tau)\,d\tau.\end{aligned} \tag{2.3}$$

Then it follows from (1.1) and (2.1)–(2.3) that

$$\begin{aligned}\int_0^t L(t,s)x(s)\,ds &= \int_0^t K(t,s)x(s)\,ds + \int_0^t \Phi_s(t,s)x(s)\,ds \\ &\quad + \int_0^t \Phi(t,s)A(s)x(s)\,ds \\ &\quad + \int_0^t \left[\int_s^t \Phi(t,u)K(u,s)\,du\right] x(s)\,ds \\ &= x'(t) - A(t)x(t) - F(t) + \Phi(t,t)x(t) \\ &\quad - \Phi(t,0)x_0 - \int_0^t \Phi(t,s)F(s)\,ds \\ &= x'(t) - B(t)x(t) + H(t).\end{aligned}$$

Thus every solution of (1.1) is also a solution of (2.1). Conversely, let $y(t)$ be a solution of (2.1) with $y(0) = x_0$. Define

$$z(t) = y'(t) - F(t) - A(t)y(t) - \int_0^t K(t,s)y(s)\,ds.$$

From (2.1), (2.2) and the definition of $z(t)$ we obtain

$$z(t) = -\Phi(t,t)y(t) + \int_0^t \left[\Phi_s(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s)\,du\right] y(s)\,ds$$
$$+ \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F(s)\,ds.$$

Substituting for $F(s)$ from the definition of $z(t)$ and changing the order of integration, we get

$$z(t) = -\left[\Phi(t,t)y(t) - \Phi(t,0)x_0 - \int_0^t \Phi(t,s)y'(s)\,ds\right]$$
$$+ \int_0^t \Phi_s(t,s)y(s)\,ds - \int_0^t \Phi(t,s)z(s)\,ds.$$

From the identity

$$\int_0^t \Phi(t,s)y'(s)\,ds = \Phi(t,t)y(t) - \Phi(t,0)x_0 - \int_0^t \Phi_s(t,s)y(s)\,ds$$

it is clear that

$$z(t) = -\int_0^t \Phi(t,s)z(s)\,ds.$$

Since $\Phi(t,s)$ is continuous, it follows from the uniqueness of solutions of Volterra integral equations that $z(t) = 0$. Hence $y(t)$ solves (1.1).

REMARK 2.2. It is to be noted that if $\Phi(t,s)$ is the differentiable resolvent corresponding to the kernel $K(t,s)$, then the equation (2.1) together with (2.2) gives the usual variation of constants formula (see Grossman and Miller [**8**]).

LEMMA 2.3. *Let $B(t)$ be an $n \times n$ continuous matrix which commutes with its integral and let M and α be positive numbers. Suppose the inequality*

$$\left|\exp\left(\int_s^t B(\tau)\,d\tau\right)\right| \le Me^{-\alpha(t-s)}, \qquad 0 \le s \le t < \infty, \tag{2.4}$$

holds. Then every solution $x(t)$ of (2.1) with $x(0) = x_0$ satisfies

$$|x(t)| \le M|x_0|e^{-\alpha t} + M\int_0^t e^{-\alpha(t-\tau)}|H(\tau)|\,d\tau \tag{2.5}$$
$$+ M\int_0^t \left[\int_s^t e^{-\alpha(t-\tau)}|L(\tau,s)|\,d\tau\right] |x(s)|\,ds.$$

PROOF. Multiplying both sides of (2.1) by $\exp(-\int_0^t B(\tau)\,d\tau)$ and rearranging the terms, we obtain

$$\left(\exp\left(-\int_0^t B(\tau)\,d\tau\right)x(t)\right)' = \exp\left(-\int_0^t B(\tau)\,d\tau\right)\left[H(t) + \int_0^t L(t,s)x(s)\,ds\right].$$

Integrating from 0 to t, we get

$$\exp\left(-\int_0^t B(\tau)\,d\tau\right)x(t) = x_0 + \int_0^t \exp\left(-\int_0^s B(\tau)\,d\tau\right)H(s)\,ds$$
$$+ \int_0^t \exp\left(-\int_0^s B(\tau)\,d\tau\right)\left(\int_0^s L(s,u)x(u)\,du\right)\,ds.$$

By changing the order of integration on the right side and using (2.4), we obtain (2.5).

REMARK 2.4. If B is a constant matrix, then it commutes with its integral. Further, the condition (2.4) holds if, in addition, all the characteristic roots of B have negative real parts (cf. [**12**, Chapter 2]).

THEOREM 2.5. *Let $\Phi(t,s)$ be a continuously differentiable $n\times n$ matrix function such that, for $0 \le s \le t < \infty$,*

(i) *the hypotheses of Lemma* 2.3 *holds,*

(ii) $|\Phi(t,s)| \le L_0 e^{-\gamma(t-s)}$,

(iii) $\sup_{0\le s\le t<\infty} \int_s^t e^{\alpha(\tau-s)}|L(\tau,s)|\,d\tau \le \alpha_0$,

where $L_0, \gamma\ (> \alpha), \alpha_0$ are positive real numbers. Suppose further

(iv) $F(t) \equiv 0$,

where $F(t)$ is defined in (1.1). *If $\alpha - M\alpha_0 > 0$, then every solution $x(t)$ of* (1.1) *tends to zero exponentially as $t \to +\infty$.*

PROOF. In view of Lemma 2.1 and the function $\Phi(t,s)$ satisfying the conditions (i), (ii) and (iii), it is enough to show that every solution of (2.1) tends to zero exponentially as $t \to +\infty$. Since $F(t) \equiv 0$, the equation (2.2) and (2.5) and the condition (ii) imply that

$$e^{\alpha t}|x(t)| \le M|x_0| + ML_0|x_0| \int_0^t e^{-(\gamma-\alpha)\tau}\,d\tau + M\int_0^t \left[\int_s^t e^{\alpha\tau}|L(\tau,s)|\,d\tau\right]|x(s)|\,ds.$$

Using (iii), we get

$$e^{\alpha t}|x(t)| \le M|x_0| + \frac{ML_0|x_0|}{(\gamma-\alpha)} + \int_0^t M\alpha_0 e^{\alpha s}|x(s)|\,ds.$$

The application of Gronwall inequality yields that

$$e^{\alpha t}|x(t)| \le M|x_0|\left(1 + \frac{L_0}{(\gamma-\alpha)}\right)e^{M\alpha_0 t}.$$

This implies that

$$|x(t)| \le M|x_0|\left(1 + \frac{L_0}{(\gamma-\alpha)}\right)e^{-(\alpha-M\alpha_0)t}.$$

Thus in view of $\alpha - M\alpha_0 > 0$, the result follows.

COROLLARY 2.6. *In addition to the assumptions* (i), (ii) *and* (iv) *of Theorem* 2.5, *suppose the following conditions hold*:

(a) $|K(t,s)| \le K_0 e^{-\beta(t-s)}$ *for* $0 \le s \le t < \infty$,

(b) $|\Phi_s(t,s)| \le N_0 e^{-\delta(t-s)}$ *for* $0 \le s \le t < \infty$,

(c) $|A(t)| \le A_0$ *for* $0 \le t < \infty$

where $A_0, N_0, K_0, \beta, \delta$ are positive real numbers, and

(d) $\gamma > \beta > \alpha$, $\delta > \alpha$ *and* $\alpha - M\hat{\alpha}_0 > 0$

where

$$\hat{\alpha}_0 \stackrel{\text{def}}{=} \left[\frac{K_0}{\beta-\alpha} + \frac{N_0}{\delta-\alpha} + \frac{L_0A_0}{\gamma-\alpha} + \frac{K_0L_0}{(\beta-\alpha)(\gamma-\beta)}\right].$$

Then every solution $x(t)$ of (1.1) *tends to zero exponentially as $t \to +\infty$.*

PROOF. Following the proof of Theorem 2.5, we obtain

$$\begin{aligned}(2.6)\quad e^{\alpha t}|x(t)| \le M|x_0|\left(1+\frac{L_0}{(\gamma-\alpha)}\right) \\ + M\int_0^t \left[\int_s^t e^{\alpha\tau}\middle| K(\tau,s)+\Phi_s(\tau,s)+\Phi(\tau,s)A(s)\right. \\ \left. + \int_s^\tau \Phi(\tau,u)K(u,s)\,du\middle|\, d\tau\right] |x(s)|\,ds.\end{aligned}$$

Using conditions (i), (a), (b), (c) and estimating each integral on the right side of (2.6), we get

$$\begin{aligned}e^{\alpha t}|x(t)| \le M|x_0|\left(1+\frac{L_0}{(\gamma-\alpha)}\right) \\ + \int_0^t M\hat{\alpha}_0 e^{\alpha s}|x(s)|\,ds.\end{aligned}$$

Thus, in view of condition (d), the application of Gronwall's inequality yields the desired result.

REMARK 2.7. If $F(t) \equiv 0$ in equation (1.1), then the Theorem 2.5 asserts that the zero solution of (1.1) is exponentially asymptotically stable.

REMARK 2.8. If $F(t)$ is not zero in Theorem 2.5, still the solutions of (1.1) tends to zero as $t \to +\infty$ provided $\int_0^\infty |F(s)|\,ds < \infty$. This is an immediate consequences of variation of constants formula (see [**8**]) and Theorem 2.5.

REMARK 2.9. It is possible to select a matrix function $\Phi(t,s)$ satisfying the conditions (i) and (ii) of Theorem 2.5 and condition (b) of Corollary 2.6. For example, if $\Phi(t,s) = L_0 e^{-\gamma(t-s)}I$, then $N_0 = L_0\gamma$ and $\delta = \gamma$. $\Phi(t,t)$ being a constant matrix in this case, the estimate (2.4) is guaranteed if $A(t)$ is a constant matrix and B is negative definite.

REMARK 2.10. Basically it is the condition "$M\hat{\alpha}_0 < \alpha$" in Corollary 2.6 which controls the asymptotic nature of the solution $x(t)$ of (1.1). A look at the composition of $\hat{\alpha}_0$ reveals that while so choosing γ and δ much away from β and α, respectively, we can nullify the effect of the last three terms in $\hat{\alpha}_0$, the first term $K_0/(\beta-\alpha)$ being the essential term which we have to reckon with. Therefore, if β is so large as to exceed $(\alpha^2 + MK_0)/\alpha$, then $M\hat{\alpha}_0$ would be less than α. Thus we see that the attenuation required on the kernel $K(t,s)$ is linked with the constant α in (2.4). This conclusion implicity assumed the estimate (2.4). Such an estimate would be possible when the transformed matrix B is constant and negative definite.

REMARK 2.11. In [**3**], a condition of the type (2.4) has been used for the matrix $Q \stackrel{\text{def}}{=} (A(t) - G(t,t))$, where $G(t,s)$ is the anti-derivative of the kernel $K(t,s)$ (i.e. $\partial G(t,s)/\partial t = K(t,s)$). As such the matrix B in our study allows more flexibility due to the arbitrary character of the function $\Phi(t,s)$. Further, our approach is entirely different and the analysis in [**3**] can be applied to equation (2.1) in order to obtain sharper estimates. Thus our Theorem 2.5 is in addition to the Theorem 2 of [**3**] rather than a substitute for it.

EXAMPLE 2.12. In (1.1) (scalar case), let $A(t) = a_1 e^{-b_1 t} - a_2$, $K(t,s) = e^{-b_2(t,s)}$ and $F(t) \equiv 0$ where a_1, a_2, b_1, b_2 are positive real numbers. Choose

$\Phi(t,s) = a_1 e^{-b_1 t}$. Then $M = 1, \alpha = a_2, A_0 = a_1 + a_2, L_0 = a_1, K_0 = 1, \beta = b_2, \gamma = \delta = b_1$ and $N_0 = 0$. Thus the condition (d) of Corollary 2.6 holds if $a_1 = \hat{K} a_2$, $b_1 = (4\hat{K}^2 + 4\hat{K} + 1)a_2, b_2 = (4 + a_2^2)/a_2$ and $a_2 \geq 2/[\hat{K}(4\hat{K}+3)]^{1/2}$ where $\hat{K}$ $(1 < \hat{K} < \infty)$ is an arbitrary real number. For example, if $\hat{K} = 2, a_2 = 1$, then $M\hat{\alpha}_0 \simeq 0.53$ and $\alpha = 1$.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, KANPUR–208016, INDIA

Current address: Department of Computer Science, University of Houston, University Park, Houston, Texas 77004