ODD STARLIKE FUNCTIONS
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Abstract. In the present paper, among other things, we prove that if \( f'(0) = 1 \) is regular and odd starlike in \( |z| < 1 \), then \( \Re f(z)/s_n(z, f) > 1/2 \), \( n = 1,2,3, \ldots \), thus generalising the known result: \( \Re f(z)/z > 1/2, |z| < 1 \). As an application, we show that each partial sum of an odd convex function is close-to-convex in \( |z| < 1 \).

Introduction. Let \( A \) denote the class of functions \( f \) which are regular in the unit disc, \( \mathbb{D} = \{ z | |z| < 1 \} \), and are normalised by \( f(0) = f'(0) - 1 = 0 \). We denote by \( S \) the subclass of \( A \) consisting of univalent functions. For a given real number \( \alpha \), \( 0 < \alpha < 1 \), let \( S_\alpha(\alpha) \) and \( K(\alpha) \) represent the subclasses of \( S \) consisting of starlike functions of order \( \alpha \) and convex functions of order \( \alpha \), respectively. \( S_\alpha(0) \) and \( K(0) \) will be simply denoted by \( S_\alpha \) and \( K \), and will, respectively, be referred to as the classes of starlike and convex functions. It is known that \( K \subset S_{\alpha}(1/2) \). Finally, let \( C \) denote the subclass of \( S \) made up of close-to-convex functions. A sufficient condition for \( f \in A \) to be in \( C \) is that \( \Re f'(z) > 0 \) in \( \mathbb{D} \).

Marx [2] and Strohhäcker [7] proved independently that if \( f \in K \), then \( \Re (f(z)/z) > 1/2 \), \( z \in \mathbb{D} \). Their result was generalised in two ways by Ruscheweyh and Sheil-Small [5] who proved that if \( f \in S_{\alpha}(1/2) \), then \( \Re (f(z)/s_n(z, f)) > 1/2, z \in \mathbb{D} \), where \( s_n(z, f) \) denotes the \( n \)th partial sum of \( f(z) \). It is also known that if \( f \in A \) and \( \Re f'(z) > \alpha, \alpha < 1 \), then \( \Re (f(z)/z) > \alpha, z \in \mathbb{D} \). For \( f \in K \), Ruscheweyh [4] determined the radius of close-to-convexity of \( s_n(z, f) \) depending upon \( n \). Recently, Robertson [3] proved that if \( f \in K(1/2) \), then all its partial sums \( s_n(z, f) \) are close-to-convex in \( \mathbb{D} \).

In the present short note we prove that if \( f \) is an odd starlike function or \( f \in A \) and satisfies the condition \( \Re f'(z) > 1/2 \) in \( \mathbb{D} \), then \( \Re (f(z)/s_n(z, f)) > 1/2, z \in \mathbb{D} \). We also show that all partial sums of odd convex functions are close-to-convex in \( \mathbb{D} \).

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are regular in \( |z| < r_1 \) and \( |z| < r_2 \), respectively, then the function \( f \ast g \), defined by the power series

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,
\]

is regular in \( |z| < r \), where \( r = \min\{r_1, r_2\} \).
is called the Hadamard product/convolution of $f$ and $g$, and is regular at least in the disc $|z| < r_1 r_2$.

We shall need the following results.

**Lemma 1.** If $\phi$ and $g$ belong to $S_\lambda(1/2)$, then for any function $F$, regular in $D$, the function

$$\frac{(\phi \ast Fg)(z)}{(\phi \ast g)(z)} \quad (z \in D),$$

takes values in the closed convex hull of $F(D)$.

**Lemma 2.** If $f \in A$ and satisfies the condition $\text{Re} f'(z) > 1/2$ in $D$, then for all points $z$ and $z_0$ in $D$, we have

$$\text{Re} \frac{f(z) - f(z_0)}{z - z_0} > \frac{1}{2}.$$

Lemma 1 is due to Ruscheweyh and Sheil-Small [5] and Lemma 2 is an easy consequence of a result of Sakaguchi [6].

We now prove the following:

**Theorem 1.** If $f(z) = z + a_3 z^3 + a_5 z^5 + \cdots$ is an odd starlike function, then for all $n \geq 1$, we have

$$\text{Re} \frac{f(z)}{s_n(z, f)} > \frac{1}{2} \quad (z \in D).$$

**Proof.** Since $f(z) = z + a_3 z^3 + a_5 z^5 + \cdots$ is an odd starlike function, it is easily verified that the function $f_1$, defined by

$$f_1(z) = \sqrt{z} f(\sqrt{z}) = z + a_3 z^2 + a_5 z^3 + \cdots + a_{2n-1} z^n + \cdots,$$

is in $S_\lambda(1/2)$.

Let $g(z) = z/(1 - z)$. Then $g \in K \subseteq S_\lambda(1/2)$. Thus both $f$ and $g$ belong to $S_\lambda(1/2)$. In view of Lemma 1, we therefore conclude that the function

$$\psi(z) = \frac{f_1(z) \ast g(z)(1 - z^n)}{f_1(z) \ast g(z)}$$

takes values in the closed convex hull of $F(D)$, where $F(z) = (1 - z^n)$. In other words,

$$|\psi(z) - 1| < 1 \quad (z \in D),$$
or,

$$\left| \frac{z + a_3 z^2 + a_5 z^3 + \cdots + a_{2n-1} z^n}{\sqrt{z} f(\sqrt{z})} - 1 \right| < 1 \quad (z \in D),$$
or,

$$\left| \frac{s_n(\sqrt{z}, f)}{f(\sqrt{z})} - 1 \right| < 1 \quad (z \in D),$$
which implies, setting $\sqrt{z} = \xi$, that
\[
\text{Re} \frac{f(\xi)}{s_n(\xi, f)} > \frac{1}{2} \quad (\xi \in D).
\]

This completes the proof of our theorem.

**Theorem 2.** If $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ is an odd convex function, then all partial sums $s_n(z, f)$ are close-to-convex in $D$ with respect to $f$.

**Proof.** Since $f(z)$ is an odd convex function, $zf'(z)$ is odd and starlike in $D$. Hence, in view of Theorem 1, we conclude that for all $z \in D$
\[
\text{Re} \frac{zf'(z)}{s_n(z, zf')} > \frac{1}{2},
\]
or, equivalently,
\[
\text{Re} \frac{f'(z)}{s_n(z, f)} > \frac{1}{2} \quad (z \in D).
\]
The last inequality implies that $\text{Re}(s_n'(z, f)/f'(z)) > 0$ in $D$, and the desired conclusion follows.

**Theorem 3.** If $f \in A$ and satisfies the condition $\text{Re} f'(z) > 1/2$ in $D$, then for all $n \geq 1$,
\[
\text{Re} \frac{f(z)}{s_n(z, f)} > \frac{1}{2} \quad (z \in D).
\]

**Proof.** Let $f(z) = z + \sum_{n=2}^{\infty}a_nz^n$. Since $\text{Re} f'(z) > 1/2$ in $D$, in view of Lemma 2, we have for a given $z_0 \in D$,
\[
\text{Re} \frac{f(z) - f(z_0)}{z - z_0} > \frac{1}{2} \quad (z \in D),
\]
or,
\[
\text{Re} \left[ \frac{f(z_0)}{z_0} + \sum_{n=1}^{\infty} \frac{f(z_0) - s_n(z_0, f)}{z_0^{n+1}} z^n \right] > \frac{1}{2} \quad (z \in D),
\]
from which it follows that
\[
\left| \frac{f(z_0) - s_n(z_0, f)}{z_0^{n+1}} \right| \leq \text{Re} \frac{f(z_0)}{z_0} \leq \left| \frac{f(z_0)}{z_0} \right| \quad [1, \text{p. 41}].
\]

Therefore,
\[
\left| \frac{s_n(z_0, f)}{f(z_0)} - 1 \right| \leq |z_0|^n < 1,
\]
or equivalently,
\[
\text{Re} \frac{f(z_0)}{s_n(z_0, f)} > \frac{1}{2}.
\]

This completes the proof of our theorem.
References


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