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ODD STARLIKE FUNCTIONS

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ABSTRACT. In the present paper, among other things, we prove that if f ($f(0) = 0$, $f'(0) = 1$) is regular and odd starlike in $|z| < 1$, then $\operatorname{Re} f(z)/s_n(z, f) > 1/2$, $|z| < 1$, where $s_n(z, f)$ denotes the nth partial sum of f, $n = 1, 2, 3, \ldots$, thus generalising the known result: $\operatorname{Re} f(z)/z > 1/2$, $|z| < 1$. As an application, we show that each partial sum of an odd convex function is close-to-convex in $|z| < 1$.

Introduction. Let A denote the class of functions f which are regular in the unit disc, $\mathscr{D} = \{z \,|\, |z| < 1\}$, and are normalised by $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of univalent functions. For a given real number α, $0 \leqslant \alpha < 1$, let $S_t(\alpha)$ and $K(\alpha)$ represent the subclasses of S consisting of starlike functions of order α and convex functions of order α, respectively. $S_t(0)$ and $K(0)$ will be simply denoted by S_t and K, and will, respectively, be referred to as the classes of starlike and convex functions. It is known that $K \subset S_t(1/2)$. Finally, let C denote the subclass of S made up of close-to-convex functions. A sufficient condition for $f \in A$ to be in C is that $\operatorname{Re} f'(z) > 0$ in $\mathbf{D}$.

Marx [2] and Strohhäcker [7] proved independently that if $f \in K$, then $\operatorname{Re}(f(z)/z) > 1/2$, $z \in \mathbf{D}$. Their result was generalised in two ways by Ruscheweyh and Sheil-Small [5] who proved that if $f \in S_t(1/2)$, then $\operatorname{Re}(f(z)/s_n(z, f)) > 1/2$, $z \in \mathbf{D}$, where $s_n(z, f)$ denotes the nth partial sum of $f(z)$. It is also known that if $f \in A$ and $\operatorname{Re} f'(z) > \alpha$, $\alpha < 1$, then $\operatorname{Re}(f(z)/z) > \alpha$, $z \in \mathbf{D}$. For $f \in K$, Ruscheweyh [4] determined the radius of close-to-convexity of $s_n(z, f)$ depending upon n. Recently, Robertson [3] proved that if $f \in K(1/2)$, then all its partial sums $s_n(z, f)$ are close-to-convex in $\mathbf{D}$.

In the present short note we prove that if f is an odd starlike function or $f \in A$ and satisfies the condition $\operatorname{Re} f'(z) > 1/2$ in $\mathbf{D}$, then $\operatorname{Re}(f(z)/s_n(z, f)) > 1/2$, $z \in \mathbf{D}$. We also show that all partial sums of odd convex functions are close-to-convex in $\mathbf{D}$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are regular in $|z| < r_1$ and $|z| < r_2$, respectively, then the function $f * g$, defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

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is called the Hadamard product/convolution of f and g, and is regular at least in the disc $|z| < r_1 r_2$.

We shall need the following results.

LEMMA 1. *If ϕ and g belong to $S_t(1/2)$, then for any function F, regular in* **D**, *the function*

$$\frac{(\phi * Fg)(z)}{(\phi * g)(z)} \qquad (z \in \mathbf{D}),$$

takes values in the closed convex hull of $F(\mathbf{D})$.

LEMMA 2. *If $f \in A$ and satisfies the condition $\operatorname{Re} f'(z) > 1/2$ in* **D**, *then for all points z and z_0 in* **D**, *we have*

$$\operatorname{Re} \frac{f(z) - f(z_0)}{z - z_0} > \frac{1}{2}.$$

Lemma 1 is due to Ruscheweyh and Sheil-Small **[5]** and Lemma 2 is an easy consequence of a result of Sakaguchi **[6]**.

We now prove the following:

THEOREM 1. *If $f(z) = z + a_3 z^3 + a_5 z^5 + \cdots$ is an odd starlike function, then for all $n \geqslant 1$, we have*

$$\operatorname{Re} \frac{f(z)}{s_n(z, f)} > \frac{1}{2} \qquad (z \in \mathbf{D}).$$

PROOF. Since $f(z) = z + a_3 z^3 + a_5 z^5 + \cdots$ is an odd starlike function, it is easily verified that the function f_1, defined by

$$f_1(z) = \sqrt{z} f(\sqrt{z}) = z + a_3 z^2 + a_5 z^3 + \cdots + a_{2n-1} z^n + \cdots,$$

is in $S_t(1/2)$.

Let $g(z) = z/(1 - z)$. Then $g \in K \subset S_t(1/2)$. Thus both f and g belong to $S_t(1/2)$. In view of Lemma 1, we therefore conclude that the function

$$\psi(z) = \frac{f_1(z) * g(z)(1 - z^n)}{f_1(z) * g(z)}$$

takes values in the closed convex hull of $F(\mathbf{D})$, where $F(z) = (1 - z^n)$. In other words,

$$|\psi(z) - 1| < 1 \qquad (z \in \mathbf{D}),$$

or,

$$\left| \frac{z + a_3 z^2 + a_5 z^3 + \cdots + a_{2n-1} z^n}{\sqrt{z} f(\sqrt{z})} - 1 \right| < 1 \qquad (z \in \mathbf{D}),$$

or,

$$\left| \frac{s_n(\sqrt{z}, f)}{f(\sqrt{z})} - 1 \right| < 1 \qquad (z \in \mathbf{D}),$$

which implies, setting $\sqrt{z} = \xi$, that

$$\operatorname{Re}\frac{f(\xi)}{s_n(\xi, f)} > \frac{1}{2} \qquad (\xi \in \mathbf{D}).$$

This completes the proof of our theorem.

THEOREM 2. *If $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ is an odd convex function, then all partial sums $s_n(z, f)$ are close-to-convex in* **D** *with respect to f.*

PROOF. Since $f(z)$ is an odd convex function, $zf'(z)$ is odd and starlike in **D**. Hence, in view of Theorem 1, we conclude that for all $z \in \mathbf{D}$

$$\operatorname{Re}\frac{zf'(z)}{s_n(z, zf')} > \frac{1}{2},$$

or, equivalently,

$$\operatorname{Re}\frac{f'(z)}{s_n'(z, f)} > \frac{1}{2} \qquad (z \in \mathbf{D}).$$

The last inequality implies that $\operatorname{Re}(s_n'(z, f)/f'(z)) > 0$ in **D**, and the desired conclusion follows.

THEOREM 3. *If $f \in A$ and satisfies the condition* $\operatorname{Re} f'(z) > 1/2$ *in* **D**, *then for all $n \geqslant 1$,*

$$\operatorname{Re}\frac{f(z)}{s_n(z, f)} > \frac{1}{2} \qquad (z \in \mathbf{D}).$$

PROOF. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Since $\operatorname{Re} f'(z) > 1/2$ in **D**, in view of Lemma 2, we have for a given $z_0 \in \mathbf{D}$,

$$\operatorname{Re}\frac{f(z) - f(z_0)}{z - z_0} > \frac{1}{2} \qquad (z \in \mathbf{D}),$$

or,

$$\operatorname{Re}\left[\frac{f(z_0)}{z_0} + \sum_{n=1}^{\infty} \frac{f(z_0) - s_n(z_0, f)}{z_0^{n+1}} z^n\right] > \frac{1}{2} \qquad (z \in \mathbf{D}),$$

from which it follows that

$$\left|\frac{f(z_0) - s_n(z_0, f)}{z_0^{n+1}}\right| \leqslant \operatorname{Re}\frac{f(z_0)}{z_0} \leqslant \left|\frac{f(z_0)}{z_0}\right| \qquad [\mathbf{1}, \text{p. } 41].$$

Therefore,

$$\left|\frac{s_n(z_0, f)}{f(z_0)} - 1\right| \leqslant |z_0|^n < 1,$$

or equivalently,

$$\operatorname{Re}\frac{f(z_0)}{s_n(z_0, f)} > \frac{1}{2}.$$

This completes the proof of our theorem.

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