ON THE CONVOLUTION EQUATIONS IN THE SPACE OF DISTRIBUTIONS OF $L^p$-GROWTH

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Abstract. We consider convolution equations in the space $D'_p$, $1 \leq p \leq \infty$, of distributions of $L^p$-growth, i.e., distributions which are finite sums of derivatives of $L^p$-functions (see [4, 7]). Our main results are to find a condition for convolution operators to be hypoelliptic in $D'_p$ in terms of their Fourier transforms and to show that the same condition is working for the solvability of convolution operators in the tempered distribution space $\mathcal{S}'$ and $D'_p$.

Preliminary. We recall the basic facts about the spaces $\mathcal{D}'_p$, $1 \leq p \leq \infty$, and $\mathcal{S}'$, which we need in this paper. For the proof we refer to [4, 7].

The space $\mathcal{D}'_p$, $1 \leq p \leq \infty$. Let $\mathcal{D}'_p$, be the space of all $C^\infty$-functions $\phi$ in $\mathbb{R}^n$ such that $\mathcal{D}'_p$, for all $\alpha \in \mathbb{N}^n$, is in $L^p(\mathbb{R}^n)$ equipped with the topology generated by countable norms

$$\|\phi\|_{m,p} = \left\{ \sum_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^p}^p \right\}^{1/2}, \quad m \in \mathbb{N}, 1 \leq p < \infty,$$

and

$$\|\phi\|_{m,\infty} = \sup_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^\infty}, \quad m \in \mathbb{N}.$$

Then it is obviously a Fréchet space and a normal space of distributions in $\mathbb{R}^n$. We also have $C^\infty \subset \mathcal{D}'_p \subset \mathcal{D}'$ with continuous injections.

We denote by $\mathcal{D}'_p$, $1 \leq p \leq \infty$, the dual of $\mathcal{D}'_q$, where $1/p + 1/q = 1$ and these duals are subspaces of the space of distributions in $\mathbb{R}^n$. A distribution $T$ is in $\mathcal{D}'_p$, $1 \leq p \leq \infty$, if and only if there is an integer $m(T) > 0$ such that

$$T = \sum_{|\alpha| \leq m} D^\alpha f_\alpha, \quad \alpha \in \mathbb{N}^n,$$

where the $f_\alpha$'s are bounded continuous functions belonging to $L^p(\mathbb{R}^n)$. Moreover, if $p < \infty$, each $f_\alpha$ converges to zero at infinity.

The Fourier transform of a function in $\mathcal{D}'_p$, is a continuous function rapidly decreasing at infinity and also the Fourier transform of a distribution in $\mathcal{D}'_p$, is a continuous function slowly increasing at infinity.
The space $\mathcal{S}'$. Let $\mathcal{S}$ be the space of all $C^\infty$-functions $\phi$ in $\mathbb{R}^n$ such that
\[
\sup_{|\alpha| \leq k, x \in \mathbb{R}^n} (1 + |x|)^k |D^\alpha \phi(x)| < \infty, \quad k = 0, 1, 2, \ldots,
\]
equipped with the topology generated by these countable norms. We denote by $\mathcal{S}'$ the dual of $\mathcal{S}$. The Fourier transformation is now an isomorphism of $\mathcal{S}$ onto itself and of $\mathcal{S}'$ onto $\mathcal{S}'$.

The space $\mathcal{E}'(\mathcal{S}' : \mathcal{S}')$ of convolution operators in $\mathcal{S}'$ consists of distributions $S \in \mathcal{S}'$ satisfying one of the following equivalent conditions:

(i) Given any $k = 1, 2, \ldots$, $S$ can be represented in the form
\[
S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha,
\]
where $f_\alpha, |\alpha| \leq m$, are continuous functions in $\mathbb{R}^n$ such that
\[
f_\alpha(x) = O\left((1 + |x|)^{-k}\right) \quad \text{as} \ |x| \to \infty.
\]

(ii) For every $\phi$ in $\mathcal{S}$, $S \ast \phi$ is in $\mathcal{S}$. Moreover, the mapping $\phi \to S \ast \phi$ of $\mathcal{S}$ into $\mathcal{S}$ is continuous.

The Fourier transform $\hat{S}$ of a distribution $S$ in $\mathcal{E}'(\mathcal{S}' : \mathcal{S}')$ is a $C^\infty$-function with the following property: For every multi-index $\alpha$ there exists a nonnegative integer $l$ such that
\[
D^\alpha \hat{S}(\xi) = O\left((1 + |\xi|)^{l}\right) \quad \text{as} \ |\xi| \to \infty.
\]

We denote by $\mathcal{E}_M(\mathcal{S}' : \mathcal{S}')$ the space of all $C^\infty$-functions with the above property (3). They are multiplication operators in $\mathcal{S}'$ and the Fourier transformation is an isomorphism of $\mathcal{E}_M(\mathcal{S}' : \mathcal{S}')$ onto $\mathcal{E}_M(\mathcal{S}' : \mathcal{S}')$ (see [7, Volume II]).

Hypoelliptic convolution equations in the space $\mathcal{D}'_{L^p}$, $1 < p < \infty$. In [10], Zielézny showed how to define, in a general manner, hypoelliptic and entire elliptic convolution operators in subspace of the space of distributions. He also characterized hypoelliptic and entire elliptic convolution operators in the space $\mathcal{S}'$ of tempered distributions. In [6 and 12], he studied hypoelliptic convolution operators in the space of distributions of exponential growth of polynomial power and, in [5], Paik studied the same problem in the space of distributions of generalized exponential growth introduced in [2].

In this paper it can be seen that for a distribution $S$ in $\mathcal{E}_M$ the hypoellipticity of the convolution operator $S$ in the space of tempered distributions is equivalent to the hypoellipticity in the space of bounded distributions. We define hypoelliptic convolution operators in $\mathcal{D}_{L^\infty}$ as follows: A distribution $S$ in $\mathcal{D}_{L^1}$ is said to be hypoelliptic in $\mathcal{D}_{L^\infty}$, if every solution $U$ in $\mathcal{D}_{L^\infty}$ of the convolution equation
\[
S \ast U = V
\]
is in $\mathcal{D}_{L^\infty}$, when $V$ is in $\mathcal{D}_{L^\infty}$; in that case equation (1) is also called hypoelliptic in $\mathcal{D}_{L^\infty}$. Since the space of convolution operators in $\mathcal{D}_{L^\infty}$ is $\mathcal{D}_{L^1}$, hypoelliptic convolution operators in $\mathcal{D}_{L^\infty}$ has to be characterized in $\mathcal{D}_{L^1}$. Because of lack of differentiability of their Fourier transforms there are some difficulties to achieve our goal. In
this paper we only consider subclasses of $\mathcal{D}'_\omega$, containing $\mathcal{O}'$, whose Fourier transforms have certain order derivatives and increase slowly at infinity. In this class we can characterize hypoelliptic convolution operators in $\mathcal{D}'_\omega$. But we have an example of hypoelliptic convolution operators in $\mathcal{D}'_\omega$ which is not in this class.

We now establish a necessary and sufficient condition for a convolution operator to be hypoelliptic in $\mathcal{D}'_\omega$. The result is proved only for a subclass of convolution operators in $\mathcal{D}'_\omega$ and the proof is based on an idea similar to that used in [10 and 12].

**Definition.** $S \in \mathcal{D}'_\omega$ is said to be of class $H_m$ if the Fourier transform $\hat{S}$ is a $C^m$-function in $\mathbb{R}^n$ and $D^\alpha S$, $|\alpha| \leq m$, are slowly increasing at infinity.

The fact that the Fourier transform is a topological isomorphism from $\mathcal{O}'$ onto $\mathcal{O}_M$ (see [1, Chapter VII]) implies that every distribution in $\mathcal{O}'$ is of class $H_m$. This class $H_m$ of distributions in $\mathcal{D}'_\omega$ will be used for our study of hypoellipticity in $\mathcal{D}'_\omega$. We begin with a lemma.

**Lemma.** Let $S$ be a distribution whose Fourier transform is of the form

$$\hat{S} = \sum_{j=1}^{\infty} a_j \delta(\xi_j),$$

where the $\xi_j$ satisfy the condition

$$|\xi_j| > 2|\xi_{j-1}| > 2^j, \quad j = 1, 2, \ldots,$$

and the $a_j$ are complex numbers such that

$$|a_j| = O\left(|\xi_j|^\mu\right) \quad \text{as } j \to \infty$$

for some $\mu$; then the series in (5) converges in $\mathcal{D}'_\omega$. We assert that $S \in \mathcal{D}_o$ if and only if

$$|a_j| = O\left(|\xi_j|^{-\nu}\right) \quad \text{as } j \to \infty$$

for every $\nu > 0$.

**Proof.** Using the fact that, for $\phi \in \mathcal{D}_\omega$,

$$|\xi^\alpha \hat{\phi}(\xi)| \leq \|D^\alpha \phi\|_{\omega}, \quad \alpha \in \mathbb{N}^n,$$

the Fourier transforms of functions in a bounded set in $\mathcal{D}_\omega$ are uniformly $O(|\xi|^{-\nu})$ as $|\xi| \to \infty$, for every $\nu > 0$. Therefore the series $S = \sum_{j=1}^\infty a_j e^{i(x \cdot \xi_j)}$ converges in $\mathcal{D}'_\omega$. If the $a_j$ satisfy the condition (8), then the last series and all its term-by-term derivatives converge uniformly in $\mathbb{R}^n$. Consequently, $S$ is a $C^\infty$-function bounded together with its derivatives and so belongs to $\mathcal{D}_\omega$. The converse proof is exactly the same in [10].

We are now in a position to prove our main theorem.

**Theorem 1.** Let $S$ be a distribution in $\mathcal{D}'_\omega$ which is of class $H_m$, $m > n$. Then $S$ is hypoelliptic in $\mathcal{D}'_\omega$ if and only if its Fourier transform satisfies the following condition: There are constants $a$ and $M$ such that

$$|\hat{S}(\xi)| \geq |\xi|^a \quad \text{for } \xi \in \mathbb{R}^n \text{ and } |\xi| \geq M.$$
Proof. Suppose that the condition (9) is not satisfied. Then there exists a sequence \( \xi_j \) in \( \mathbb{R}^n \) defined as in the Lemma and such that

\[
|\hat{S}(\xi_j)| < |\xi_j|^{-j}, \quad j = 1, 2, \ldots.
\]

Then the series

\[
U = \sum_{j=1}^{\infty} e^{i\langle x, \xi_j \rangle}
\]

converges in \( \mathcal{D}'_\mathbb{R} \), but by the Lemma \( U \) is not in \( \mathcal{D}_{L^\infty} \). On the other hand,

\[
S \ast U = \sum_{j=1}^{\infty} \hat{S}(\xi_j) e^{i\langle x, \xi_j \rangle},
\]

and applying the Lemma we conclude that \( S \ast U \) is in \( \mathcal{D}_{L^\infty} \). Thus \( S \) is not hypoelliptic in \( \mathcal{D}'_{L^\infty} \).

Conversely, let us take a \( C^\infty \)-function \( \psi \) in \( \mathbb{R}^n \) such that

\[
\psi(\xi) = \begin{cases} 
1 & \text{for } |\xi| < M, \\
0 & \text{for } |\xi| > M + 1,
\end{cases}
\]

where \( M \) is the constant in (9). Then we define the Fourier transform \( \hat{P} \) of \( P \) by the formula

\[
\hat{P}(\xi) = \begin{cases} 
0 & \text{for } |\xi| < M, \\
\frac{1 - \psi(\xi)}{\hat{S}(\xi)} & \text{for } |\xi| \geq M.
\end{cases}
\]

Obviously \( \hat{S} \) is a \( C^m \)-function slowly increasing together with its derivatives up to the \( m \)th order. From the fact that \( S \) is of class \( H_m \) and (9) we can choose a positive integer \( k \) so large that

\[
\mathcal{D}^k(\xi), |\alpha| \leq m, \text{ are in } L^1(\mathbb{R}^n) \text{ and vanish at infinity}, \text{ which follows from the iterated "chain rule"}
\]

\[
\partial^\alpha \left( \frac{1}{\hat{S}} \right) = \sum \frac{\pi^{\alpha_k} \partial^{\alpha_k} \hat{S}}{\hat{S}^{k+1}} C_{\alpha_1 \cdots \alpha_k}, \quad \alpha_1 + \cdots + \alpha_k = \alpha.
\]

Then we have, applying integration by parts,

\[
|Q(x)| = \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \hat{Q}(\xi) \, d\xi \right|
\]

\[
= \left| \frac{1}{(2\pi)^{n/2}} \frac{1}{(1 + |x|^2)^{m/2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} (1 - \Delta)^{m/2} \hat{Q}(\xi) \, d\xi \right|
\]

\[
< C \frac{1}{(1 + |x|^2)^{m/2}} \text{ for some constant } C.
\]
Therefore $Q(x)$ is an $L^1$-function, and so the distribution
\[ P = (1 + D_1^2 + \cdots + D_n^2)^k Q(x) \]
is in $\mathcal{D}'$. Furthermore $\hat{S}(\xi)\hat{P}(\xi) = 1 - \psi(\xi)$, whence, passing to the inverse Fourier transform, we see that $P$ is a rapidly decreasing parametrix for $S$ with $\hat{W} = \psi$, that is,
\[ S \ast P = \delta - W. \]
(12)

Now assume that $S \ast U = V$, where $V \in \mathcal{D}_{L\infty}$ and $U \in \mathcal{D}'_{L\infty}$. Then, making use of (12), we can write
\[ U = U \ast \delta = U \ast (S \ast P) + U \ast W \]
\[ = (U \ast S) \ast P + U \ast W \]
\[ = V \ast P + U \ast W. \]

It is obvious that $V \ast P$ and $U \ast W$ are in $\mathcal{D}_{L\infty}$, so that $U$ is in $\mathcal{D}_{L\infty}$.

**COROLLARY.** With the same hypothesis of $S$ in the theorem, (9) implies that every solution $U$ in $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, of the equation (4) is in $\mathcal{D}_{L^\infty}$ whenever $V$ is in $\mathcal{D}_{L^p}$.

**PROOF.** Viewing the proof of sufficiency of the theorem, $P$ is in $\mathcal{D}'_{L^1}$ and $U = V \ast P + U \ast W$. We can easily see the $\mathcal{D}_{L^p} \ast \mathcal{D}'_{L^1} \subset \mathcal{D}_{L^p}$ and so $U$ is in $\mathcal{D}_{L^p}$.

If the given convolution operator $S$ is in $\mathcal{D}'_{L^1}$, then we have the following weak version of the regularity theorem.

**THEOREM 2.** If a distribution $S$ in $D'_{L^1}$ satisfies the condition (9), then every solution $U$ in $\mathcal{D}'_{L^\infty}$ of the equation (1) with $V \in \mathcal{D}_{L^1}$ is in $\mathcal{D}_{L^\infty}$.

**PROOF.** Applying the same argument as in Theorem 1, we construct the continuous function $\hat{P}(\xi)$ slowly increasing at infinity, and so we find a positive integer $k$ so large that
\[ \hat{Q}(\xi) = \frac{1}{(1 + |\xi|^2)^k} \hat{\psi}(\xi) \]
is in $L^2(\mathbb{R}^n)$. By Plancherel's theorem, $Q(x)$ is in $L^2(\mathbb{R}^n)$, and so the distribution $P = (1 + D_1^2 + \cdots + D_n^2)^k Q$ is in $\mathcal{D}'_{L^2}$. Also, we have
\[ U = U \ast \delta = V \ast P + U \ast W. \]

Since $V$ is in $\mathcal{D}_{L^2}$, $V \ast P$ and $U \ast W$ are in $\mathcal{D}_{L^\infty}$, so that $U$ is in $\mathcal{D}_{L^\infty}$.

Combining Theorem 1 with the results of [10] we can state

**THEOREM 3.** Let $S$ be a distribution in $\mathcal{C}'_{L^1}$. Then the following are equivalent:
(a) $S$ is hypoelliptic in $\mathcal{D}'$.
(b) $S$ is hypoelliptic in $\mathcal{D}'_{L^\infty}$.
(c) There exist constants $\alpha$ and $M$ such that
\[ |\hat{S}(\xi)| \geq |\xi|^\alpha \text{ for } \xi \in \mathbb{R}^n \text{ and } |\xi| \geq M. \]

We now give two examples of hypoelliptic convolution operators, one of which is not of class $H_m$. 

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Example 1. Let $S = e^{-|\xi|}$ in $\mathbb{R}^1$. Since $\hat{S}(\xi) = 1/(1 + \xi^2)$, it is in $\mathcal{S}'$ and satisfies the condition (9). Therefore, it is hypoelliptic in $\mathcal{S}'$ and $\mathcal{D}'_{L^\infty}$.

Example 2. Taking $S = 1/(1 + x^2) + \delta$ in $\mathbb{R}^1$, $\hat{S}(\xi) = e^{-|\xi|} + 1$ is not a $C^1$-function in $\mathbb{R}^1$ and satisfies the condition (9) in Theorem 1 with $a = -1$ and $M = 1$. From the fact that $1/(1 + x^2)$ is in $\mathcal{D}_L^1$, and $\mathcal{D}'_{L^\infty} \subset \mathcal{D}'_{L^\infty}$, $S$ is a hypoelliptic convolution operator in $\mathcal{D}'_{L^\infty}$.

Remark. We can easily see that the convolution operator $S$ in $\mathcal{S}'$ is solvable in $\mathcal{S}'$, if and only if $\hat{S}$ satisfies the property (9) and has no zero in $\mathbb{R}^n$ and also characterizes the solvability in $\mathcal{D}'_L$ because $S$ is actually invertible in $\mathcal{S}'$. Therefore, the convolution operators in $\mathcal{S}'$ which have the above properties are solvable in the spaces $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, and $\mathcal{S}'$, but the converse does not hold in general. We still leave the problems such as what condition guarantees the hypoellipticity in $\mathcal{D}'_{L^\infty}$ for the general convolutors and the solvability in $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, and $\mathcal{S}'$ without invertibility.

Comment. We appreciate the referee for various suggestions to reform our paper. He also suggested to study the relation between the number $m$ in the condition $(H_m)$ and the number $p$ which the Fourier transform of $\hat{S}(\xi)^{-1}(1 + |\xi|^2)^{-k}$, for sufficiently large $k$, is an $L^p$-convolutor.

References


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